Fundamentals of Electric Theory and Circuits

by

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Charge Densities and Continuity in Conductors and Semiconductors and The propagation of Electromagnetic signals in straight wires

Number references in square brackets [ ] *are listed in "*Fundamentals of Electric Theory and Circuits". Where not explicitly indicated, Chapter and Section references are in the textbook "Fundamentals of Electric Theory and Circuits"
Number references (*superscript*) *are listed at the end of the article.*

The article is divided in 2 sections.

The first section discusses charge densities in conductors and semiconductors generally and charge flow. The conservation of charge principle is applied to derive the equation of continuity when charges flow.

In the second section we will derive the equation describing the flow of a variable current in straight wires from Newton`s second law of motion plus Weber`s electro-dynamics. This gives accurate information on the surface and volume components of charge distribution and charge movement during signal propagation in dc, ac and transient situations. The derivation shows how in special situations electromagnetic signals can propagate at light speeds.

At the end of the section we discuss the limitations of the Maxwell electrodynamics (classical electrodynamics) especially with slowly varying effects. Finally, we discuss how the electromagnetic field can be derived from Weber's action–at–a–distance theory and how this field may be extended to rapidly varying effects and radiation by introducing time retardation.

1. Surface and Volume Charge Densities and Continuity in Conductors and Semiconductors

In Chapter 2, Section 2.21 we saw how excess charge on an insulated spherical conductor spreads uniformly around to reach static equilibrium. And in Chapter 1, Section 1.1 we learned that there cannot be a volume charge inside a conductor in the steady-state and that all excess charge resides on the surface. The examples relate to situations in electrostatics and electric circuits (in the dc steady-state).

In courses of electric circuits, transmission lines and electronic circuits, one frequently encounters different situations which involve visualizing the formation, motion and decay of volume and surface charge densities. Is the process steady as in the wires of a simple dc electrical circuit? Is the process transient prior to reaching a steady-state? Or is it a quasistatic situation? Are the charges "excess" or are they the charges of the material itself or are the charges termed "excess" when they have migrated from other portions of the closed system comprising a circuit and all its elements?

The time it takes for volume charge densities to decay in conductors in isolation and circuits is called the *relaxation* time, a time which is of the order of 10^{-14} seconds.

The analysis of such transient and quasistatic phenomena requires making macroscopic connections to microscopic processes.

We begin by reviewing a few definitions and principles involved in current and the flow of charge and then explore the relations (continuity equations) between charge density and current density functions in various circuit operation situations.

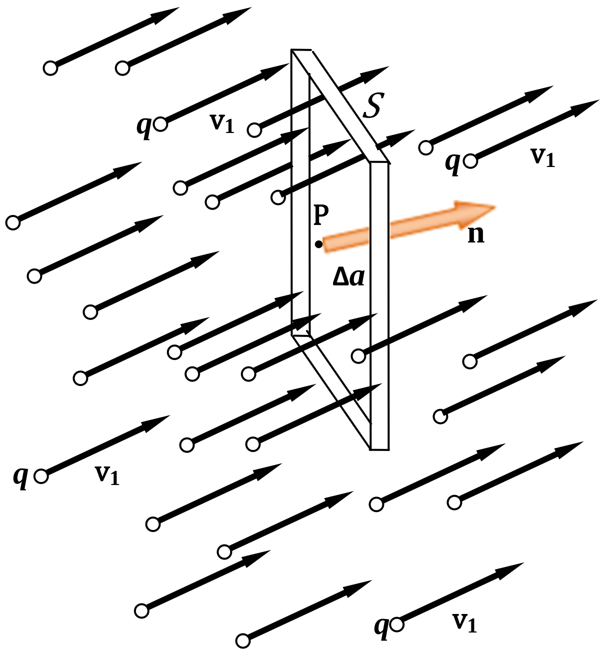
Conservation of electric charge

The conservation of electric charge requires that the net charge is conserved under all conditions; that is the algebraic sum of the charges within a closed system is constant. Mathematically, $\sum_j q_j = \text{constant}$ for any closed system.

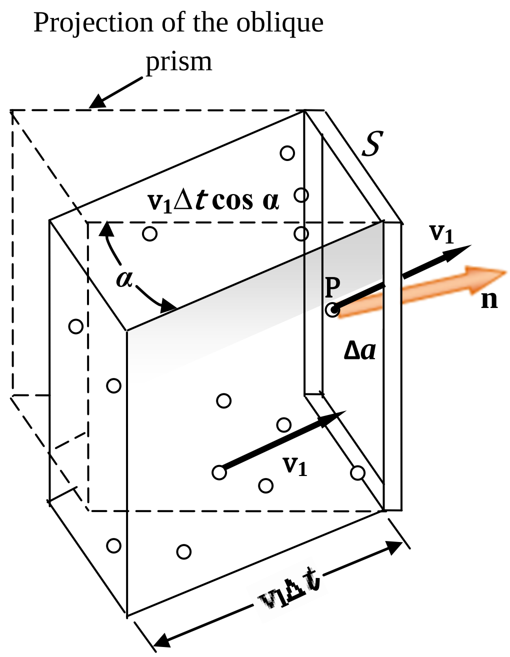
Current I and Current Density J

We usually consider current along a well-defined path, like a wire. If the current is steady – that is, unchanging in time – it must be the same at every point along the wire, just as with steady traffic the same number of cars must pass, per hour, different points along an unbranching road. A more general kind of current, or charge transport, involves charge carriers moving around in three-dimensional space. To describe this we need the concept of *current density*.

Consider a current across a small plane surface S of area Δa (see Fig.(a)) fixed in some orientation, whose unit normal is **n** and whose direction is considered as that of a current flow called positive across S. (The unit normal vector is the most elegant way to describe the direction of a thin patch of surface and the reader is urged to take a small rectangular patch of paper and slide it all over the outer surface of a curved bottle or a curved glass flask and note how the direction of the normal on the paper best represents the direction of a tiny patch on continuously curved surface areas).



(a)



(b)

There is a velocity $\mathbf{v_1}$ of all the charges of a certain kind (electrons, say) in the near vicinity of *S* when Δ*a* becomes sufficiently small in all its dimensions i.e. shrinks about a point P in Δ*a*. It may not be that each charge has the same velocity as every other charge near P, keeping in mind that in general, in conductors there are millions and millions of charges having a random movement (see Chapter 1, Section 1.17). So, the velocity $\mathbf{v_1}$ in general may mean an average over all charges in a neighborhood of P at a given time.

How many particles pass through the frame in a time interval Δt?

The net charge (maybe millions of electrons, say) crossing *S* in an infinitesimal time interval Δ*t*, due to the motion of carriers whose charge is q_1 and whose number per unit volume is n_1, is the total charge of the carriers contained in the oblique prism Fig. (b).

If Δ*t* begins at the instant shown in Figs. (a) and (b), the particles destined to pass through the frame in the next Δ*t* sec will be just those now located within the oblique prism in Fig.(b). This prism has the face area Δ*a* as its base, an edge length $v_1\Delta t$ which is the distance any particle will travel in a time Δ*t*. Particles outside the prism will either miss the window or fail to reach it.

The volume of the oblique prism *dV* is the product base × altitude or $(v_1\Delta t)$ Δ*a* cos α, where α is the angle between $\mathbf{v_1}$ and **n**. A rectangular prism with the slanting edge dimension $(v_1\Delta t)$ cos α as its edge length is shown in the figure by projecting the oblique prism. Briefly in vector notation

$$dV = \mathbf{v}_1 \cdot \mathbf{n}\, \Delta t\, \Delta a \qquad (1)$$

The infinitesimal charge crossing the surface is

$dq = n_1 q_1 dV$ (where n_1 is the no. of charge carriers per unit volume) (2)

The current across S is therefore

$$\Delta I = n_1 q_1 \mathbf{v}_1 \cdot \mathbf{n}\, \Delta a \quad (3)$$

where ΔI is the symbol used since the current is that which crosses Δa. This equation becomes exact when Δa approaches zero by shrinking about the point P. So, in differential form (where Algebra ends and Calculus begins!)

$$dI = n_1 q_1 \mathbf{v}_1 \cdot \mathbf{n}\, da \quad (4)$$

The product

$$\mathbf{J} = n_1 q_1 \mathbf{v}_1 \quad (5)$$

which appears in Eq. (5) is the vector which we shall call the current density. Substitution of (5) in (4) gives

$$dI = \mathbf{J} \cdot \mathbf{n}\, da \quad (6)$$

Equation (6) is more general than Eq. (4) and is applicable for example, when more than one type of charged particle participates in the current flow, as is the case with electrolytic solutions. In that case, Eq. (5) must then be generalized to

$$\mathbf{J} = \sum_i n_i q_i \mathbf{v}_i \quad (7)$$

Eq.(6) may be obtained from a picture in which no discrete charges appear at all, but instead a charge density ρ (also sometimes called the **volume charge density**) is conceived to move with a velocity **v** being functions of the coordinates and perhaps of the time.

Since the product $n_1 q_1$ of Eq. (5) is a charge density when viewed macroscopically, it is easy to see that in terms of moving charge density ρ_1 we may write in place of Eq. (5)

$$\mathbf{J} = \rho_1 \mathbf{v}_1 \quad (8)$$

The validity of Eq. (8) rests upon the conservation of electric charge and upon the physical postulate that the velocity function $\mathbf{v}_i$ may be considered continuous either for particles or for charge density. That the validity of Eq. (8) is established is important because it also establishes the validity of the interior of current carrying wires to be neutral in the steady-state as will be discussed soon. From the above postulates it follows that the current across an arbitrary infinitesimal surface depends upon the orientation of the surface in the manner expressed by Eq. (6). This equation may be considered to be the defining equation for the current density vector. Eqns. (7), of which Eq. (5) is a special case and Eq. (8) are then simply relations between **J** and the velocities that produce it. Eq. (6) may be integrated to yield the total current across a finite surface S (which is not closed but open)

$$I = \int_S \mathbf{J} \cdot \mathbf{n}\, da \qquad (9)$$

The neutrality of a current carrying wire

It may seem from the postulates of the previous section that the charged particles or charge density exist with a net macroscopic charge density as the flow of electrons in between the electrodes of a vacuum diode or of a klystron.

In the Appendix A we gave a proof that the interior of a current carrying wire in the steady-state is neutral. In that situation what was flowing? It should be noted carefully that current density can perfectly well exist in the absence of any net macroscopic charge density. There may be both positive (lattice ions) and negative elementary particles present in a conductor in such numbers that the resulting net charge in every macroscopic element of volume vanishes.

If charges of both sign are free to migrate simultaneously, they make contributions of the same sign to current density, since the sense of $\mathbf{v}_i$ reverses with the sign of q_i.

The Equation of Continuity- a general form

The conservation of electric charge requires that the charge density and current density functions be related(Electricity and Magnetism by Edson Ruther Peck, McGraw Hill, 1953). Either may exist without the other (excess point charge placed on an insulated conducting sphere and static equilibrium is reached (Chapter 2, Section 2.21), and the current in the interior of a wire (Chapter 1, Section 1.18)), or both may exist together (during the time when the excess point charges placed migrate to spread on the surface of an insulated conducting sphere to reach static equilibrium). But, this does not mean that they are quite independent of one another.

We can derive a relation between them for systems where the volume density of charge ρ and the current density J are finite and together completely describe the charge and current of the system. The functions are assumed to be continuous, so that any macroscopic discontinuities are treated simply as regions of rapid change (say, at the boundary of two dissimilar metals).

We will then discuss three useful applications of the principle of conservation of charge to obtain the equations of continuity for a current carrying wire, the surface current of a cylindrical conductor, and for a *p*-type semiconductor.

Consider a volume V enclosed by a fixed surface S and let **n** be the unit normal to S everywhere out from V. If currents are flowing, there may be a net rate of change of charge within V, for in general the net current in or out of V will not vanish.

An expression for this net current is obtained at once from Eq. (9). Considering the current positive if outward from V (an example of how one should define the direction of a current), we have

$$I = \oint_S \mathbf{J} \cdot \mathbf{n}\, da \tag{10}$$

where the integral symbol used represents that the surface over which the integration is to be carried out is "closed". From the definition of current, I is the rate at which charge flows out from V across S; and by conservation of charge, I is the rate at which the net charge q in V is decreasing (*no charge can flow away from a place without diminishing the amount of charge that is there*). Thus

$$I = -\frac{dq}{dt} \tag{11}$$

so that

$$\oint_S \mathbf{J} \cdot \mathbf{n}\, da + \frac{dq}{dt} = 0 \tag{12}$$

For the reader who has done a course in DC circuit theory, Eq. (11) may come as a bit of a surprise because current is conserved all around in a DC circuit in the steady-state. Our derivation of the continuity equation is more general and includes situations when there may be a local generation and/or neutralization of charged particles. We will gain more insight when we discuss the continuity equations for conductors and semiconductors.

Now, q may be expressed in terms of charge density ρ:

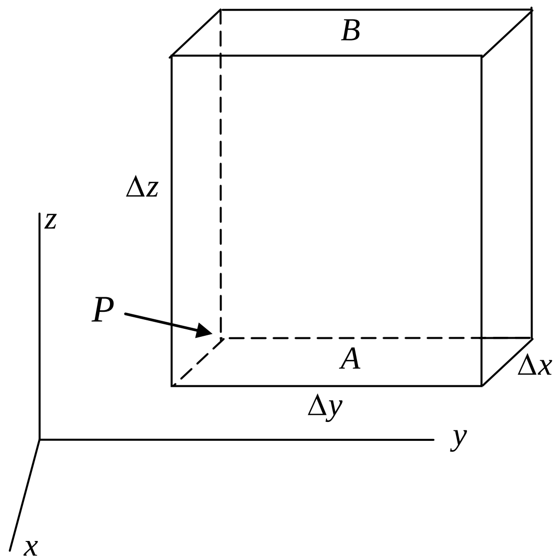
$$q = \int_V \rho\, dv \tag{13}$$

In this way, we are led to an integral equation connecting J and ρ:

$$\oint_S \mathbf{J} \cdot \mathbf{n}\, da + \int_V \frac{\partial \rho}{\partial t}\, dv = 0 \tag{14}$$

This integral equation Eq. (14) holds for any arbitrary volume V with its enclosing surface S, and requires the existence of a differential relation between **J** and ρ which holds at every point of space.

We now obtain a **differential relation** between **J** and ρ. Consider a small rectangular box whose edges are parallel to the coordinate axes and whose dimensions are Δx, Δy and Δz. This box, is to be only a mathematical construction made at some arbitrary point of the system P(*x*,*y*,*z*), which is taken as a corner point of the box; and for convenience, let P be that corner point having lowest algebraic values of *x*, *y*, and *z*.



Consider the current out from the box through the two opposite faces *A* and *B* which are parallel to the *xy* plane. Only the *z* component of the current density will contribute to this current; a positive *z* component of current density gives an inward current through *A* and an outward current through *B*. Let a general point on face *A* have coordinates ($x + \alpha\,\Delta x$, $y + \beta\,\Delta y$, z), and let a corresponding point on *B* have coordinates ($x + \alpha\,\Delta x$, $y + \beta\,\Delta y$, $z + \Delta z$). Here α and β are fractions running from zero to unity (e.g. 10 slices of 0.1 or, 100 slices of 0.01), and they will serve as the variables in the computation of the current from the two faces *A* and *B* of the box. The values of (*x*,*y*,*z*), and of (Δx,Δy,Δz) will be considered fixed during the computations.

We specify that the box shall be small, so that the current density over the faces *A* and *B* may be expressed by a Taylor's expansion in the variables α and β. This expansion is, for the *z* component of the current density, as follows. On *A*,

$$(J_z)_A = J_z(x,y,z) + \left(\frac{\partial J_z}{\partial x}\right)_{x\,y\,z} \alpha\,\Delta x + \left(\frac{\partial J_z}{\partial y}\right)_{x\,y\,z} \beta\,\Delta y$$

$$+ \text{ quadratic and higher-order terms in } (\alpha\,\Delta x) \text{ and } (\beta\,\Delta y) \qquad (15)$$

$$(J_z)_B = J_z(x,y,z) + \left(\frac{\partial J_z}{\partial x}\right)_{x\,y\,z} \alpha\,\Delta x + \left(\frac{\partial J_z}{\partial y}\right)_{x\,y\,z} \beta\,\Delta y + \left(\frac{\partial J_z}{\partial z}\right)_{x\,y\,z} \Delta z$$

$$+ \text{ quadratic and higher-order terms in } (\alpha\,\Delta x), (\beta\,\Delta y) \text{ and } \Delta z \qquad (16)$$

We have used the first-order terms of the expansion, that is, to remember that the current density is a function of position, in order to get a nonzero answer for the net current from the box; but we shall neglect quadratic and higher-order terms, assuming that Δ*x*, Δ*y*, and Δ*z* are to be considered as small as necessary so that the remainder of the series expansion becomes as small as we please relative to the first-order terms.

We now use these expansions to compute the current through faces *A* and *B*:

$$I_B = \int_B (J_z)_B \, dS \qquad I_A = \int_A (J_z)_A \, dS \tag{17}$$

Now the element of area on the faces of *A* and *B* has the form

$$dS = d(\alpha \, \Delta x) \, d(\beta \, \Delta y) = \Delta x \, \Delta y \, d\alpha \, d\beta \tag{18}$$

Therefore, by a substitution of $(J_z)_B$ from Eq. (16) in Eq. (17) and using Eq. (18)

$$I_B = J_z(x,y,z)\Delta x \, \Delta y \int_0^1 \int_0^1 d\alpha \, d\beta + \left(\frac{\partial J_z}{\partial x}\right)_{x,y,z} (\Delta x)^2 \Delta y \int_0^1 \int_0^1 \alpha \, d\alpha \, d\beta + \left(\frac{\partial J_z}{\partial y}\right)_{x,y,z} \Delta x (\Delta y)^2 \int_0^1 \int_0^1 \beta \, d\alpha \, d\beta + \left(\frac{\partial J_z}{\partial z}\right)_{x,y,z} \Delta x \, \Delta y \, \Delta z \int_0^1 \int_0^1 d\alpha \, d\beta \tag{19}$$

where all factors which are constant during the integration have been taken outside the integral signs. These integrals have the values 1, ½, ½, and 1, respectively because $\int_0^1 d\alpha = [\alpha + C]_0^1 = 1$, and similarly $\int_0^1 d\beta = [\beta + C]_0^1 = 1$, and $\int_0^1 \alpha d\alpha = \left[\frac{\alpha^2}{2} + C\right]_0^1 = 1/2$, and similarly $\int_0^1 \beta d\beta = \left[\frac{\beta^2}{2} + C\right]_0^1 = 1/2$.

Thus the current out through *B* is

$$I_B = J_z \Delta x \, \Delta y + \frac{1}{2}\left(\frac{\partial J_z}{\partial x}\right)(\Delta x)^2 \Delta y + \frac{1}{2}\left(\frac{\partial J_z}{\partial y}\right)\Delta x (\Delta y)^2 + \frac{1}{2}\left(\frac{\partial J_z}{\partial z}\right)\Delta x \, \Delta y \, \Delta z \tag{20}$$

in which the highly explicit notation has been simplified.

By a similar process the current into the box through face *A* is:

$$I_A = J_z \Delta x \, \Delta y + \frac{1}{2}\left(\frac{\partial J_z}{\partial x}\right)(\Delta x)^2 \Delta y + \frac{1}{2}\left(\frac{\partial J_z}{\partial y}\right)\Delta x (\Delta y)^2 \tag{21}$$

which is the same as I_B except that it lacks the last term. Thus the net current out from the box through the pair of faces *A* and *B* is

$$I_{AB} = I_B - I_A = \left(\frac{\partial J_z}{\partial z}\right)\Delta x \, \Delta y \, \Delta z \tag{22}$$

By a similar process the net current out of the box through the other two pairs of parallel faces may be obtained. The symmetry of expressions (Eq. (22)) shows that the results will be

$$\left(\frac{\partial J_x}{\partial x}\right) \Delta x\, \Delta y\, \Delta z \qquad\qquad \left(\frac{\partial J_y}{\partial y}\right) \Delta x\, \Delta y\, \Delta z \qquad (23)$$

The entire current out is therefore

$$I = \left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}\right) \Delta x\, \Delta y\, \Delta z \qquad (24)$$

This current is at the expense of a decrease of the net charge in the volume V which was expressed by Eq. (11) which we repeat for convenience.

$$I = -\frac{dq}{dt} \qquad (11)$$

If the volume density of free charge is $\rho(x,y,z)$ in the system, the net charge enclosed by the box may be found by a volume integration over the volume ΔV of the box:

$$q = \int_{\Delta V} \rho\, dV \qquad (25)$$

The integration may be performed by making a Taylor's expansion of the ρ function. The labor involved is superfluous, here, because the limiting value of the integral as Δx, Δy, and Δz approach zero is simply that due to the zero-order term in the Taylor's expansion, and may be written down at once:

$$q = \rho(x,y,z)\, \Delta V = \rho(x,y,z)\, \Delta x\, \Delta y\, \Delta z \qquad (26)$$

Therefore,

$$-\frac{dq}{dt} = -\frac{\partial}{\partial t} \rho(x,y,z)\, \Delta x\, \Delta y\, \Delta z \qquad (27)$$

where we have converted the total differential into a partial differential because the volume charge density varies over both time and space.

The current out of the volume V (Eq. 24) is equal to the rate of decrease of charge q in the volume (Eq. 27):

$$\left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}\right) \Delta x\, \Delta y\, \Delta z = -\frac{\partial}{\partial t} \rho(x,y,z)\, \Delta x\, \Delta y\, \Delta z \qquad (28)$$

Therefore,

$$\left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}\right) + \frac{\partial \rho}{\partial t} = 0 \qquad (29)$$

This is called the equation of continuity. In the abbreviated notation of vector calculus, it is written

$$\text{div. } \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \qquad (30)$$

which is a differential relation between **J** and ρ at each and every point in the conductor.

Charge within a conductor in isolation

Consider the case of the spread of excess point charge on a spherical conductor (Chapter 2, Section 2.21). We learned in Chapter 1, Section 1.14 that all net electric charge on a conductor resides on the surface of the conducting material and in Chapter 5 Section 5.22 that the time they take to dissipate is called the *relaxation* time. This statement remains true for a *homogeneous* (consisting of parts or regions similar to each other; for example, uniform structure and density of a metal), *isotropic* (whose physical properties have the same value when measured in different directions), *linear* (the current always passes in the same manner between two portions of its surface also called the 'electrodes' and whose resistance may be expressed by Ohm's Law) material even when currents are flowing.

Such a material is characterized by (Chap. 1, Sect. 1.17 and Note in Chap. 2, Sect. 2.15)

$$\mathbf{J} = \sigma \mathbf{E} \qquad (31)$$

and, (Chapter 2, Section 2.27)

$$\mathbf{D} = \varepsilon \mathbf{E} \qquad (32)$$

where σ and ε are constants. The differential equation for the electric displacement **D** is (Chapter "Electrostatics", Ref.[16]):

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho \qquad (33)$$

This equation is combined with the equation of continuity (Eq. (29)

$$\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} + \frac{\partial \rho}{\partial t} = 0 \qquad (34)$$

Using Eqs. (32) and (31) in Eqs. (33) and (34), respectively, we obtain

$$\rho = \epsilon \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) \qquad (35)$$

and

$$\frac{\partial \rho}{\partial t} = -\sigma\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) \qquad (36)$$

Hence the charge density function ρ satisfies the partial differential equation

$$\frac{\partial \rho}{\partial t} = -\frac{\sigma}{\epsilon}\rho \qquad (37)$$

The general solution of this equation is

$$\rho = \rho_o(x, y, z)\mathrm{e}^{-\left(\frac{\sigma}{\epsilon}\right)t} \qquad (38)$$

where $\rho_o(x, y, z)$ is an arbitrary distribution of charge density supposed to exist in the material at time $t = 0$.

It is important to note from the equation (38) that such a supposed charge distribution dies out at all points (note the x, y and z variables in the equation) within the conductor at a common rate $\left(\frac{\sigma}{\epsilon}\right)$ as t increases, and asymptotically approaches zero.

The quantity $\frac{\sigma}{\epsilon}$ is called the relaxation time of the material and is a convenient measure of the speed with which any charges initially in the material would die out.

If the volume charge density (excess) within a homogeneous, isotropic, linear conducting material can only decay, as in isolated metallic conductors and semiconductors, it must actually be zero there:

$$\rho = 0 \qquad (39)$$

Nothing in this theorem forbids the permanent existence of surface density of point charge on conductor boundaries or on an interface between two conducting materials (see Chapter 2, Section 2.15); for the equations used in the proof assumes continuity of the functions involved, and thus are valid only within the body of the material.

In the situation described in Chapter 2, Section 2.21 when "excess" charge ρ_o (which means that the charge is an excess over the charge density of the millions and millions of conduction band electrons of the conductive sphere) is placed on the conductive sphere, then electric fields are set up within that cause the charges to migrate towards the surface, and since the resistance of the sphere has a finite value (not zero), there will be dissipative power losses until final static equilibrium conditions prevail; no excess charge within the conductive sphere except on its surface and no electric field within, but the sphere will acquire a potential and an electric field surrounds it.

The "excess" point charge will remain on the surface unless and until it were to be discharged by connecting the sphere to the earth, or it were to discharge to the atmosphere by the phenomenon described in Chapter 1, Section 1.25.

We will now discuss the continuity equation obtained from the conservation of charge principle usually encountered in conductors, wires and semiconductors in circuits.

The continuity equation for a steady current in a DC circuit

We had devoted almost the entire Chapter 1 to a discussion of the conduction processes in simple electric circuits. It was established that the electric field in a current carrying conductor is due to surface (point) charges and also interface (point) charges (see Chapter 2, Section 2.15) at the boundaries of wires and resistors or, in general between two conductors of different resistivities.

Professor Rainer Muller in the paper “A semi quantitative treatment of surface charges in DC circuits”, Am. J. Phys., Vol. 80, No. 9, September 2012, pp 782-788, identifies two types of surface charges; one called **Type-I** which occur at the boundary of two conductors with different resistivities and the other **Type-II** which reside at the surface of conductors which sit at the boundary between a conductor and the surrounding medium (usually air).

In the important note below Fig. 1.23 it was stated that the surface charges are maintained by drifting charges due to the source of the emf. The surface and interface charges are “excess” charges, small in number in comparison to the mobile charge densities of the material of conductors and resistors. It is the surface charge which produces a uniform electric field within the conductors, wires and resistors in simple dc circuits. The analysis of the flow of these small numbers of surface and interface point charges is fairly complex and we will not make such an analysis. The steady current in a simple dc circuit is the flow of the millions of conduction band electrons already existing in the wires and resistors. These circuit elements have a volume charge density at all times. They are not “excess” and form one part of the neutral system of the conducting structure; the other part being the lattice ions.

When switched ON, it is the volume charge density which moves with a drift velocity in every elemental volume of the conducting wires and resistors, and when steady-state is reached, during this flow of the conduction band electrons, the interiors of the wires and resistors remain neutral despite the driving electric field and the current it produces. What this means is that ideally approximated, there is no “excess” charge present anywhere though the current is a start-stop motion of millions and millions of conduction band electrons everywhere within with a drift superimposed (Chapter 1, Section 1.18); no volume V(interior) is emptied of charge at any instant during the steady-state. What about the surface charges. It seems likely that the surface charges got there carried by some local “excess” volume charge drifts. However, the excess charges are “local” to the system of the circuit comprising the source of emf say, a battery, the wires and the resistors. When the circuit is switched OFF, the surface charges will reunite with the charges on the emf source taking a volume charge density drifting migratory route.

Consider the current density **J** in an isotropic medium as given by the relation Eq. (31)

$$\mathbf{J} = \sigma \mathbf{E} \qquad (31)$$

Consider the case that the material is linear and homogeneous (say, a copper or tungsten wire), in which case the conductivity σ is constant, at least by subregions. For such a material, the “excess” volume charge density is zero; and this being so at all times, the rate of change of charge density also vanishes (ignoring the small drifting charges that comprise the static surface charge provided the circuit geometry is not altered by twisting it when in operation) at all points inside the conducting material:

$$\frac{\partial \rho}{\partial t} = 0 \qquad (40)$$

The equation of continuity (Eq. (29), (remembering that this relates the charge and current density functions)

$$\left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}\right) + \frac{\partial \rho}{\partial t} = 0 \qquad (29)$$

then shows that the current density function in the region satisfies a modified differential form of the equation of continuity

$$\left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}\right) = 0 \qquad (41)$$

What Eq. (40) indicates is that for every volume charge $\Delta\rho$ leaving every elemental volume V across its enclosing surface in Δt there is a corresponding equal volume charge $\Delta\rho$ entering it in Δt, so $\Delta\rho - \Delta\rho = 0$ or, $\frac{\Delta\rho}{\Delta t} - \frac{\Delta\rho}{\Delta t} = 0$. Another way to state it is that there is no unpaired charge density (lattice ion and conduction electron). In the *absence* of emf in a region in the circuit (say, away from the source or battery and within a small section of the conductor or a resistor), the total electric field **E**, may be expressed in terms of a scalar potential function U;

$$E_x = -\frac{\partial U}{\partial x} \qquad E_y = -\frac{\partial U}{\partial y} \qquad E_z = -\frac{\partial U}{\partial z} \qquad (42)$$

Eqs. (42), (31) and (41) characterize the current flow within a region of a homogeneous, linear, isotropic conductor where there is no emf. If a dc circuit of a battery and a wire is laid in a straight line along the x-axis then evidently, the presence of surface charges will guarantee that the total field E will be a constant 'E_x' along the axis in the region. Therefore, the solution of Eq. (41) gives J_x = a constant, so using Eq. 31, we get

$$J_x = \sigma E_x = I/A \qquad (43)$$

where σ is the conductivity of the wire, I is the current in the circuit and A the cross-sectional area of the wire.

When combined, Eqs. (42), (31) and (41) also show that the potential function in such a region satisfies Laplace's equation[2, and Electricity and Magnetism by Edson Ruther Peck, McGraw-Hill, 1953]:

$$\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) = 0 \qquad (44)$$

This is the same equation as holds for a region of an electrostatic system (which is seldom described in textbooks on circuit theory) when there is no charge density and where the electric inductive capacity ε is constant. I urge the reader to study how in a conductor carrying a steady current, Laplace's equation (44) is true by referring to Chapter 1, Fig. 1.35, Ref.[8 (David J. Griffiths' Introduction to Electrodynamics) Chapter 2 Sections 2.3.1 and 2.3.3 and Chapter 3, Sections 3.1.1 and 3.1.2 with *V* in place of *U*] making note that in the interior of the neutral system of lattice ions and conduction electrons of the conductor, $\rho_{\text{excess}} = 0$.

This is how the current in the simple circuit with a battery and a bulb is described as a "disk shaped section of the electron sea that has moved in every section of the wire" in Chapter 2, Section 2.20. The flow of the charges (or current) we have described is essentially an *idealized continuous approximation* where the disk shaped section of the electron sea moves everywhere in the region without creating pockets of "local excess" volume charge densities.

And we had shown using Gauss's Law in Appendix A that the interior of a current-carrying wire in the steady-state is neutral. Once the potential function *U* is known for a given system, the problem is evidently solved, for Eqns. (42) and (31) together then yield the current density function of the mobile conduction band electrons.

What about the very initial transient? At circuit turn ON and circuit turn OFF? Will there not be an excess volume charge density?

Yes, there will be an excess volume charge density which carries the charges that will migrate to the surface (as rings of charge for round conductor); but the transient will not last long and will eventually die out when the steady-state is reached.

A similar situation will occur when the applied voltage is changed and when thermal equilibrium is disturbed caused by changes in ambient temperature. In these transient phases, $\rho_{\text{excess}} \neq 0$ and $\frac{\partial \rho_{excess}}{\partial t} \neq 0$; eventually, however the excess charge density will die away.

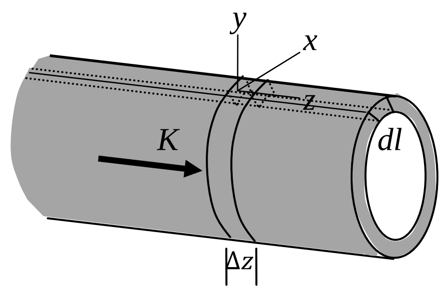
Does current vary with distance in a dc circuit?

In general, in simple dc circuits of a single loop there is no variation of current with distance in the steady-state if one were to ignore the tiny drifting current that may be associated with the very initial transient when volume charge densities may be present to supply the surface charges which will establish the electric fields in conductors and resistors. Drifting currents also occur when the circuit is switched OFF.

The continuity equation for the surface current of a cylindrical conductor

The surface current density *K* is by definition

$K \equiv \frac{dI}{dl_\perp}$ where $dl_\perp$ is a ribbon of infinitesimal width running parallel to the flow of current *dI*. The current *I* is on the surface of a cylindrical conductor or in a cylindrical shell either of radius a. In words, K is the *current per unit width* (Coulombs/m-sec) perpendicular to flow.



In particular if the surface charge density is σ_f in Coulombs/m^2 and its velocity **v** in m/s, then

$K = \sigma_f v$. (in Coulombs/m-sec) where the subscript 'f ' (for σ) denotes free or excess charge.

Now, the total current I is

$I = K \times$ width $= K \times 2\pi a$ (the circumference of the cylinder) $= 2\pi a \sigma_f v$ (in Coulombs/sec) (45)

The conservation of charge in the general form gave the equation of continuity (remembering that this relates the charge and current density functions) as div. **J** + $\frac{\partial \rho}{\partial t} = 0$ according to Eq. (30) or, equivalently $\nabla \cdot \vec{J} = -\partial \rho_f / \partial t$).

Following the logic used to derive the general form of the equation using Eqs. (24), (26) and (27) in the section "The Equation of Continuity- a general form" with $\frac{\partial I}{\partial z} \Delta z$ replacing $\left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}\right) \Delta x\, \Delta y\, \Delta z$ and $2\pi a \sigma_f(z, t) \Delta z$ replacing $\rho(x,y,z)\ \Delta x\ \Delta y\ \Delta z$, keeping in mind that there cannot be a flow of charge in the x and y ($\Delta y \rightarrow 0$) directions being a surface current, we can write

$$I = \frac{\partial I}{\partial z} \Delta z \qquad (46)$$

and

$$-\frac{dq}{dt} = -\frac{\partial (2\pi a) \sigma_f}{\partial t} \Delta z \qquad (47)$$

because the current is the time rate of surface charge flow out from an area Δz wide (annular) and the neutrality condition does not apply to the surface charge unlike in the interior.

Therefore from (46) and (47) we can write the continuity equation

$$\frac{\partial I}{\partial z} = -2\pi a \frac{\partial \sigma_f}{\partial t} \qquad (48)$$

for a cylindrical conductor with a surface current density.

Continuity conditions in transient, quasistatic and sinusoidal steady-state currents of resistive, capacitive and inductive circuits

In Chapter 2, Section 2.6 we saw the evolution of fields and surface charges of an RC circuit. In the first 160 picoseconds when the discharge of a charged capacitor was initiated, the electric field vectors cause a movement of charges to the surface of the wires. Since the field vectors point in several directions during this period before the quasistatic discharging condition, volume and surface charge densities exist in the wire when the excess charge is conveyed to and dispersed from the surface; this means that the continuity equation Eq. 41 is not satisfied and the more general continuity Eq. 29 is applicable.

In general, for time-varying signals including sinusoidal signal (steady-state) propagation in resistive, capacitive and inductive circuits and transmission lines, the continuity equation Eq. 41 will not be satisfied mostly because of the processes of accumulation and dispersion of surface charges.

In waveguides (Chapter 5, Section 5.22, Fig. 5.36(a) and Ref.[16]), surface charges appear by tiny current delivery mechanisms which is an excess volume charge density movement from the interior of the conducting plane.

In Antennas, currents of volume and surface charge densities are ever flowing in the rods which was described by Hertz as a chain of Hertzian "dipoles" when adjacent charges do not completely cancel (volume charge density movement) and there is an accumulation of charge on the surface of the wire (see See Chapter 10, Section 10.10, The Hertzian (or Ideal) Dipole).

Continuity equation for a *p*-type semiconductor

When the conservation of charge principle is applied to a p – type semiconductor, the situation is different from that of a system comprising batteries, wires and resistors or simple electric circuits.

In this we will see that the current in the volume V is due to a decrease or increase of the number of carriers in a short interval of time and $\frac{dq}{dt} \neq 0$, therefore, $\frac{\partial \rho}{\partial t} \neq 0$ and there is a distance dependent change also due to the process of diffusion.

We have to remember that in a semiconductor, there is a continuous generation and recombination of mobile carriers at each and every point within the semiconductor (see Chapter 9, Sections 9.1 and 9.2).

It is assumed that the reader is familiar with the two types of carriers in a semiconductor: *n*- and *p*- type and with the processes of generation, recombination, diffusion, and drift in semiconductors (see Chapter 9 and Appendix E and also Ref.[17, 30]).

Before we discuss the derivation of the expression of continuity in a *p*-type semiconductor, we examine the nature of current and its definition.

The electric current across a surface *S* in space having a finite or infinite area is defined as the rate at which electric charge moves across *S* and is given by Eq. (11):

$$I = -\frac{dq}{dt} \qquad (11)$$

Here the symbol *I* stands for the electric current, while *dq* is the infinitesimal amount of electric charge that has crossed *S* in the infinitesimal time interval *dt*. In this definition we have pictured *dq*/*dt* as the limit $\lim_{\Delta t \to 0} \Delta q/\Delta t$, where Δq is the finite increment of charge crossing *S* in time Δt.

One may equally well consider the form *dq*/*dt* as the derivative of a function *q*(*t*) expressing the total algebraic quantity of charge which has crossed *S* in time *t*. It is this form of the current definition which will be useful in our application of the conservation of charge principle to a semiconductor.

The definition of Eq. (11) is strictly speaking that of instantaneous current, although this adjective is usually omitted except in description of a-c circuits where special terms are to be defined anyway.

An average current across *S* would be simply

$$I_{AV} \equiv \frac{\Delta q}{\Delta t} \qquad (49)$$

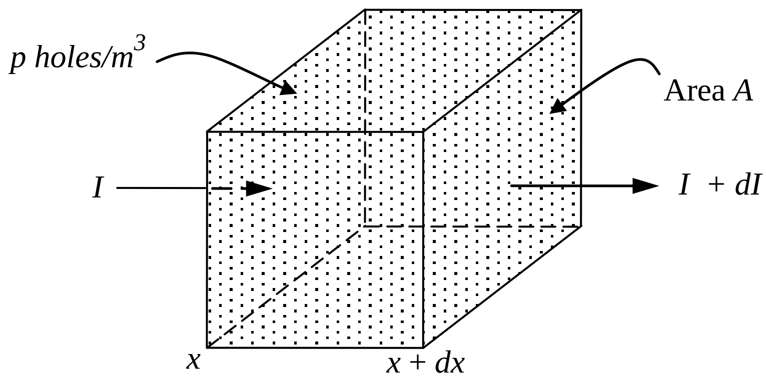
for the finite interval of time Δt.

If we disturb the equilibrium concentrations of carriers in a semiconductor material, the concentration of holes or electrons will vary with time. In the general case, however, the carrier concentration in the body of the semiconductor is a function of both time *and* distance.

This is due to the three processes of generation, recombination and diffusion which occur in semiconductor devices. Generation and recombination are time dependent processes while diffusion is a carrier concentration gradient and distance dependent process.

We will now apply the principle of conservation of charge to obtain the continuity equation (remembering that this relates the charge and current density functions) of a semiconductor (Ref.[17]).

Consider the infinitesimal element of volume of area A and length dx within which the average hole concentration is p.



If τ_p is the mean lifetime of the holes, then p/τ_p equals the holes per second lost by recombination per unit volume. If e is the electronic charge, then, because of recombination, the number of coulombs per second

$$\textit{Decreases} \text{ within the volume} = eA\,dx\,\frac{p}{\tau_p} \qquad (50)$$

If g is the thermal rate of generation of hole-electron pairs per unit volume, the number of coulombs per second

$$\textit{Increases} \text{ within the volume} = eA\,dx\,g \qquad (51)$$

Due to the process of diffusion in a carrier concentration gradient and drift in the presence of an electric field, in general, the current will vary with distance within the semiconductor.

If as indicated in the figure, the current entering the volume at x is I and leaving at $x + dx$ is $I + dI$, the number of coulombs per second

$$\textit{Decreases} \text{ within the volume} = dI \qquad (52)$$

The hole current itself is a combination of the processes of drift and diffusion which we will soon account.

Due to the three effects of recombination (Eq. (50)), generation (Eq. (51)), diffusion and drift (Eq. (52)), the hole density must change with time, and the total number of coulombs per second

$$\textit{Increases} \text{ within the volume} = eA\,dx\,\frac{dp}{dt} \qquad (53)$$

Since charge must be conserved,

$$eA\,dx\,\frac{dp}{dt} \quad = \quad -eA\,dx\,\frac{p}{\tau_p} \quad + \quad eA\,dx\,g \quad - \quad dI \qquad (54)$$

Rate of hole buildup	Rate of recombination	Rate of generation	Variation of current with distance

The hole current is the sum of the diffusion current and the drift current (the reader may refer to Appendix E and also Ref.[17]), or

$$I = -AeD_p\,\frac{dp}{dx} + \,Ape\,\mu_p E \qquad (55)$$

where D_p is the diffusion constant for holes, μ_p is the mobility for holes and **E** is the applied electric field intensity within the volume. If the semiconductor is in thermal equilibrium with its surroundings and is subjected to no applied fields, the hole density will attain a constant value p_o.

Under these conditions, $I = 0$, so that, from Eq. (54)

$$g = \frac{p_o}{\tau_p} \qquad (56)$$

The equation indicates that the rate at which holes are generated thermally just equals the rate at which holes are lost because of recombination under equilibrium conditions. Combining Eqs. (54), (55) and (56) yields the equation of *conservation of charge*, or *the continuity equation*

$$\frac{dp}{dt} = -\frac{p-p_o}{\tau_p} + \,D_p\,\frac{d^2p}{dx^2} - \,\mu_p\,\frac{d(p\mathbf{E})}{dx} \qquad (57)$$

The application of the conservation of charge principle to obtain the equation of continuity for a semiconductor may be considered the most general case where variations of carrier concentrations (volume charge densities) are seen with both time and distance even in the steady-state.

Note that the volume charge density of holes p is a function of both time and space (condition ($\frac{\partial \rho}{\partial t} \neq 0$, Eqs.(54) & (55)) unlike the case of the mobile carrier (electron) densities in metallic conductors which is uniform in both time and space in an idealized continuous approximation (condition $\frac{\partial \rho}{\partial t} = 0$, Eq.(40)).

Most textbooks discuss the solutions to the continuity equation of semiconductors (Eq. (57)) with special cases (Ref.[17]);

i) concentration independent of x with zero electric field, and

ii) concentration independent of t and with zero electric field. The details of the analysis carried out will not be discussed here.

But, the salient features of the analysis are that an "excess" concentration of holes (volume charge density) for case (i) produced by say a short dose of radiation uniform over the sample, decays with a time constant τ_p, the mean lifetime of the holes.

In case (ii), an "excess" injected concentration of holes p produced by a dose of radiation at one end of the p-type sample causes the "excess" volume charge density to decay to the equilibrium concentration p_o with a "distance" constant $L_p = \sqrt{D_p \tau_p}$.

If a p-type semiconductor sample is connected to a battery, a steady current will be established by surface and interface charges which produce a field in the sample and the holes are neutralized at the contact surfaces of the sample with the battery by electrons supplied by the battery.

Again, by a volume charge density of electron movement (start-stop motion of millions and millions of them, Drude model) within the wires connected between the battery and the semiconductor contact surface ! The motion of the electrons and holes in the semiconductor sample is also a start-stop motion of the carriers and the Drude model maybe applied to both types of carriers.

Summary

The continuity equation relates the charge and current density functions. The conservation of charge principle when applied to metallic conductors gives a continuity equation

$$\left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}\right) = 0 \quad \text{with condition } \frac{\partial \rho}{\partial t} = 0 \qquad (41)$$

from which we learn that the amount of electric charge in any volume of space can only change by the amount of electric current flowing into or out of that volume through its boundaries and the reader is cautioned to note that the equation is an idealized continuous approximation of the microscopic processes involved.

The conservation of charge principle when applied to a cylindrical conducting shell of radius a gives a continuity equation

$$\frac{\partial I}{\partial z} = -2\pi a \frac{\partial \sigma_f}{\partial t} \qquad (48)$$

where σ_f is the surface current density in Coulombs/per meter^2.

The conservation of charge principle when applied to the propagation of time-varying signals, the transient and quastatic conditions and in the sinusoidal steady-state (other than DC steady -

state situations) in resistive, capacitive and inductive circuits such as transmission lines, waveguides and in Antennas, the general form of the equation of continuity

$$\left(\frac{\partial J_x}{\partial x}+\frac{\partial J_y}{\partial y}+\frac{\partial J_z}{\partial z}\right)+\frac{\partial \rho}{\partial t}=0 \qquad (29)$$

is applicable. This is due to the accumulation and dispersion of surface charges and the processes of charge delivery and removal mechanisms that are found in such situations.

The conservation of charge principle when applied to semiconductors (p-type) gives a continuity equation

$$\frac{dp}{dt} = -\frac{p-p_o}{\tau_p} + D_p\frac{d^2p}{dx^2} - \mu_p\frac{d(p\mathbf{E})}{dx} \quad \text{with condition } \frac{\partial \rho}{\partial t}\neq 0 \qquad (57)$$

Change of carrier density over time	generation minus recombination	Difference between incoming and outgoing current

from which we learn that a change in carrier density (volume charge density) over time is due to the difference between the incoming and outgoing current plus the generation and minus the recombination.

For a very interesting electrostatic experiment with a diode and an electroscope see Chapter 3, Section 3.3.7, “Orientation of the Body Relative to the Applied Voltage” in the book “The Experimental and Historical Foundations of Electricity- Volume 2” by Andre. K. Assis. A link to the book is given in the file “Recommended_books_for_libraries” in the CD along with the book.

In the next section we discuss the propagation of electromagnetic signals in wires according to Weber’s Electrodynamics.

2. Propagation of Electromagnetic Signals in Straight Wires using Weber's Electrodynamics

Number references (*superscript*) *are listed at the end of the article.*

The theory and practice of electrical and electronic circuits involves the propagation of signals from a source to a destination. For example, a signal source which maybe a microphone, a pressure sensor, the electrode of an ECG (electro-cardio-graph) machine, or a computer sends signals through connecting wires or maybe transmission lines to a destination which maybe an amplifier, or another computer or a receiving station.

Our goal is to understand what processes enable the signals to propagate in the wires. The wires are conductors comprising lattice ions and mobile conduction band electrons and the signals are a flow of *variable* current in the wires, remembering that a steady dc current is not capable of carrying data.

From **Newton`s second law of motion plus Weber`s electrodynamics** [(1)], we derive the equation describing the flow of a variable current in straight wires and show that the signal propagates at light velocity under certain conditions.

Recall from the section "Dispelling Misconceptions about Action-at-a-distance theories" in Appendix B that Weber's Fundamental force law (Ref.[36,38]) for charged particles, is an action-at-a-distance theory, without the idea of a field. According to it, the interacting forces and energies depend only on the relative radial distance, velocity and acceleration between them. Weber's force on the line connecting the two interacting charges, obeys Newton's action and reaction law. It complies with the principles of the conservation of linear momentum, angular momentum and energy.

It would do well to remember A. K. T. Assis' statements on the validity of Webers' Electrodynamics given in Appendix B especially regarding Fechner's hypothesis. Assis has dismissed the contention of several scientists that since, Weber's force was developed from Ampere's force based on Fechner's hypothesis which states "the positive and negative charges in metallic wires move in opposite directions with equal velocities" and therefore, Weber's Electrodynamics is wrong.

Assis shows "if we assume only Weber's force and the neutrality of the current elements we can still derive Ampere's force even when Fechner's hypothesis is wrong, as is the case in the usual metallic conductors [in which the mobile charges are only the electrons]. The proof of this Section overcomes this limitation pointed out against Weber's electrodynamics as it is based on general assumptions more general than the particular case specified by Fechner's hypothesis." Assis proceeds to give the proof in Chapter 4 of "Weber's Electrodynamics" (Ref.[42]).

Weber and Kirchhoff, working independently of one another, but both utilizing Weber's electrodynamics, predicted the existence in a conducting circuit of *negligible resistance* of periodic modes of oscillation of the electric current whose velocity of propagation had the same value $c_W/\sqrt{2} = c$ as the velocity of light. This result was independent of the cross section of the wire, of its conductivity, and of the density of electricity in the wire.

Kirchhoff's work "On the motion of electricity in Wires" was published in 1857 in Poggendorff's Annalen with the English translation published in the Philosophical Magazine and Journal of Science. Another paper by Kirchhoff "On the motion of electricity in Conductors" was also published in 1857 in Poggendorff's Annalen but it was only in 1994 after a gap of nearly 137 years that it's translation by Prof. A.K.T. Assis was published. Weber's simultaneous and more thorough work was delayed in publication and appeared only in 1864 [1, Chapter 3, Section 3.1].

Assis independently published a paper in the Foundations of Physics, Vol. 30, No. 7, 2000 titled "On the Propagation of Electromagnetic Signals in Wires and Coaxial Cables According to Weber's Electrodynamics" in the year 2000 (Available: https://www.ifi.unicamp.br/~assis/Found-Phys-V30-p1107-1121(2000).pdf) which is based on Kirchhoff's papers cited above. The derivation presented below is based on Assis' paper with a few annotations and is given for the case of propagation in straight wires.

Derivation of Ampere's Force law of the force between current elements from Weber's Force Law between charge elements

It would be instructive for the reader to practice the derivation of Ampere's Force between current elements equation starting from Weber's Force law between charge elements before studying the section on the propagation of electromagnetic signals in wires. This is provided in Section 4.2 of Weber's Electrodynamics[1].

The reader should make use of the notations of Eqs. 3.10[1] and 3.14[1] while following the steps to obtain Ampere's Force equation Eq. 4.24[1] which is

$$d^2\vec{F}_{ji}^A = -\frac{\mu_o}{4\pi} I_i I_j \frac{\hat{r}_{ij}}{r_{ij}^2}\left[2\left(d\vec{I}_i \cdot d\vec{I}_j\right) - 3\left(\hat{r}_{ij} \cdot d\vec{I}_i\right)\left(\hat{r}_{ij} \cdot d\vec{I}_j\right)\right] = -d^2\vec{F}_{ji}^A$$

starting from Weber's Force Eq. 4.16[1] (see errata[1])

$$d^2\vec{F}_{ji}^W = -\frac{dq_i dq_j}{4\pi\varepsilon_o}\frac{\hat{r}}{r_{ij}^2}\left[1 + \frac{1}{c^2}\left(\vec{v}_{ij} \cdot \vec{v}_{ij}\right) - \frac{3}{2}\left(\hat{r}_{ij} \cdot \vec{v}_{ij}\right)^2 + \vec{r}_{ij} \cdot \vec{a}_{ij}\right],$$

making note of the role of Eq. 4.23[1] in converting the velocities multiplied by charge elements into current elements.

The above derivation of Ampere's Force between current elements from Weber's Force between charge elements will enable the reader understand the manner in which the velocity and acceleration of the charge have transformed into the currents in Ampere's Force law while noting the acceleration terms which exist in Eq. 4.16 do not appear in Eq. 4.24, although we did not impose any conditions on $\vec{a}_{i+}$, $\vec{a}_{i-}$, $\vec{a}_{j+}$and $\vec{a}_{j-}$. This indicates that Ampere's force remains valid even in non-stationary situations in which the charges are accelerated, not only due to the *curvature of the wires* (*centripetal accelerations*), but also when the intensity of the currents are a *function of time* (*time-varying currents*) as is the case in alternating current circuits, or when we turn on or off the current in a circuit [1, Sec. 4.2].

The most important remark is that to arrive at Ampere's force from Weber's force no conditions were imposed on $\vec{v}_{i+}$, $\vec{v}_{i-}$, $\vec{v}_{j+}$, $\vec{v}_{j-}$. These four velocities are each one of them arbitrary and independent from one another. This means that Eq. 4.24 [1] is still derived from Eq. 4.16 [1] even in metallic circuits in which the positive charges are fixed in the lattice ($\vec{v}_{i+} = 0$, $\vec{v}_{j+} = 0$) and only the moving electrons are responsible for the currents. This will also happen when the positive and negative charges move in opposite directions with velocities of different magnitudes as in situations of electrolysis, or in the usual gaseous plasma where the ratio between the velocities of the positive ions and of the electrons is as the inverse ratio of the masses[1, Sec. 4.2].

It is recommended that the reader derive Eq. 4.50[1] starting with Eq. 4.36[1] to obtain the force exerted by a closed circuit of arbitrary form on a current element of another circuit and proceed to Eq. 4.52[1]: $d\vec{F}^{A}_{C_j \; on \; I_i d\vec{l}_i} = I_i d\vec{l}_i \times \left(\frac{\mu_o}{4\pi} \oint_{C_j} \frac{I_j d\vec{l}_j \times \hat{r}_{ij}}{r_{ij}^2} \right)$

to obtain the force in terms of the magnetic field which is described by Eq. 4.75[1], which is

$$d\vec{F}_{C_2 on \; I_1 d\vec{l}_1} = I_1 d\vec{l}_1 \times \vec{B}_2 \quad \text{where} \quad \vec{B}_2 \equiv \left(\frac{\mu_o}{4\pi} \oint_{C_2} I_2 d\vec{l}_2 \times \frac{\hat{r}_{12}}{r_{12}^2} \right)$$

This will enable the reader understand the origin of the cross product in the Lorentz (magnetic) Force law.

The derivation of the equation of electromagnetic signals propagating in straight wires

There are three kinds of source charges in the wire exerting electromagnetic forces on the test one q_1: (A) Free charges over the surface of the wire, (B) the stationary positive lattice making the body of the wire, and (C) the moving conduction electrons along the body of the wire which constitute the current. We consider separately each one of them.

A) When current flows in a resistive wire connected to a battery or other power supply, the electric field driving the conduction electrons against the resistance of the wire is due to free charges distributed along the surface of the wire. This was first pointed out by Kirchhoff [2,3,4] (with English translation[5]) and further analysed by Sommerfeld[6, pp 125-130], Jefimenko[7], Heald[8], Jackson[9] and many others.

Since there cannot be zero resistive wires unless they are superconductive, in most situations in the real world there will be an electric field present to drive the conduction electrons.

The power supply or battery creates and maintains this distribution of free charges (free charges are excess charges over the combined "neutral" wire composed of lattice ions and conduction electrons). Fig. 1 shows a cylindrical wire in which an electromagnetic signal propagates and the surface charges that act on a test charge q_1.

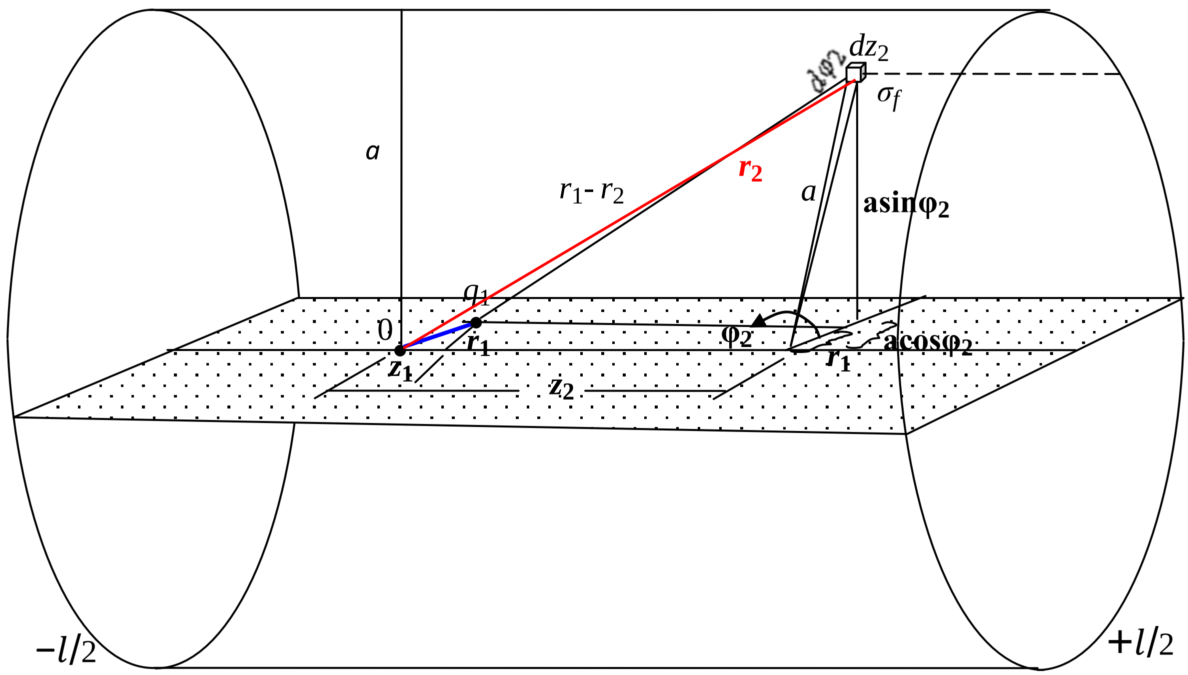


Fig. 1 (not to scale) Qualitative representation of the free surface charges with density σ_f (Coulombs/m^2) which generate the electric field inside and outside the wire. They exert a force on a test charge q_1 located at $\vec{r}_1$ and moving with velocity $\vec{v}_1$ and acceleration $\vec{a}_1$ relative to the center of the wire.

The surface density of these free (or, excess) charges is represented by $\sigma_f(z,t)$. To calculate the force exerted on the test electron q_1 by all the free charges present, we integrate the force by a charge element $dq_f = \sigma_f(z_2,t)ad\varphi_2 dz_2$ over the surface of the wire of radius a. Here φ_2 is the poloidal angle varying from 0 to 2π, $a\ d\varphi_2$ is an element of arc and dz_2 is an element of length along the wire.

B) The stationary positive lattice also exerts force on the test charge. Fig. 2 shows a charge element dq_{2+} of the lattice located at r_2 and whose volume charge density is represented by ρ_+.

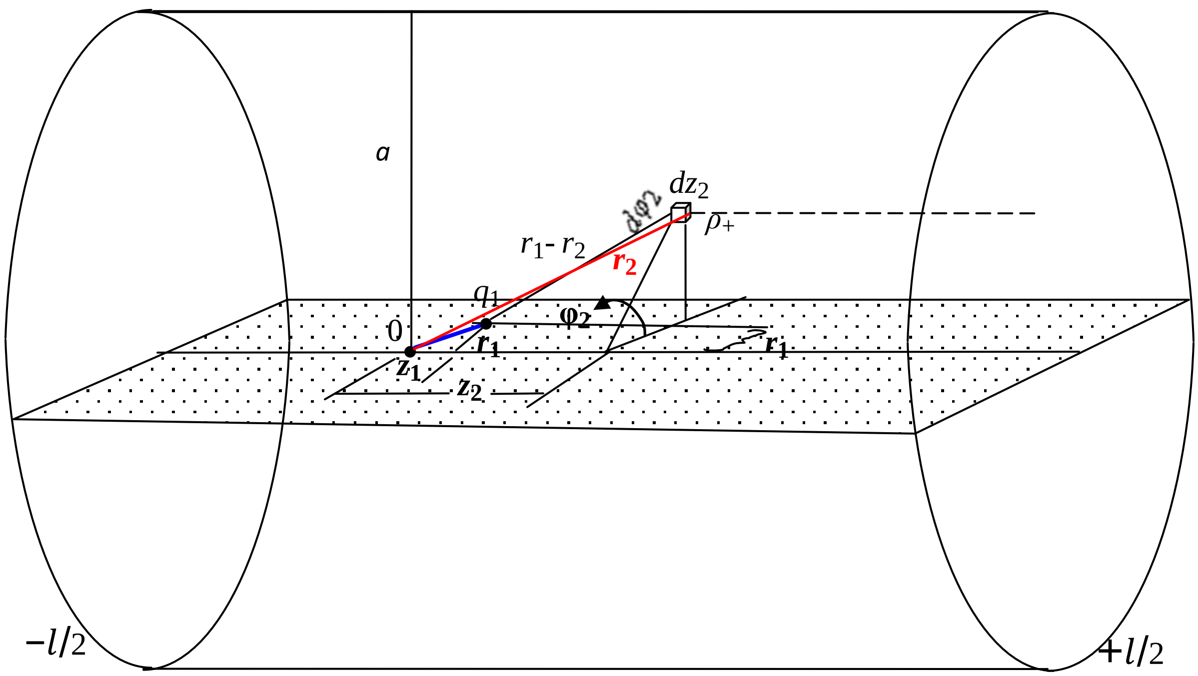


Fig. 2 (not to scale) A stationary lattice charge dq_{2+} is situated in the body of the wire located at $\vec{r}_2$.

For the homogeneous wire considered here, ρ_+ is constant over the wire and does not depend on time. We need to integrate the force on q_1 due to $dq_{2+} = \rho_+ \, r_2 \, d\varphi_2 \, dz_2$ located at a distance r_2 from the axis of the wire over its volume.

C) The moving conduction electrons also exert force on the test charge q_1. Their volume charge density for the case of Fig. 3 is represented by ρ_{c-}. For the homogeneous wires considered here ρ_{c-} is constant over the wires and does not depend on time, similar to the case of ρ_+. The velocity and acceleration of the conduction electrons at a time t in a cross section located at z_2 are represented by $\vec{v}_{2-} = v_{2-}(z_2, t)\hat{z}$ and $\vec{a}_{2-} = a_{2-}(z_2, t)\hat{z}$, respectively. We need to integrate the force on the test charge q_1 exerted by a charge element $dq_{c-} = \rho_{c-} \, r_2 \, d\varphi_2 \, dr_2 \, dz_2$ located at a distance r_2 from the axis of the wire over its volume.

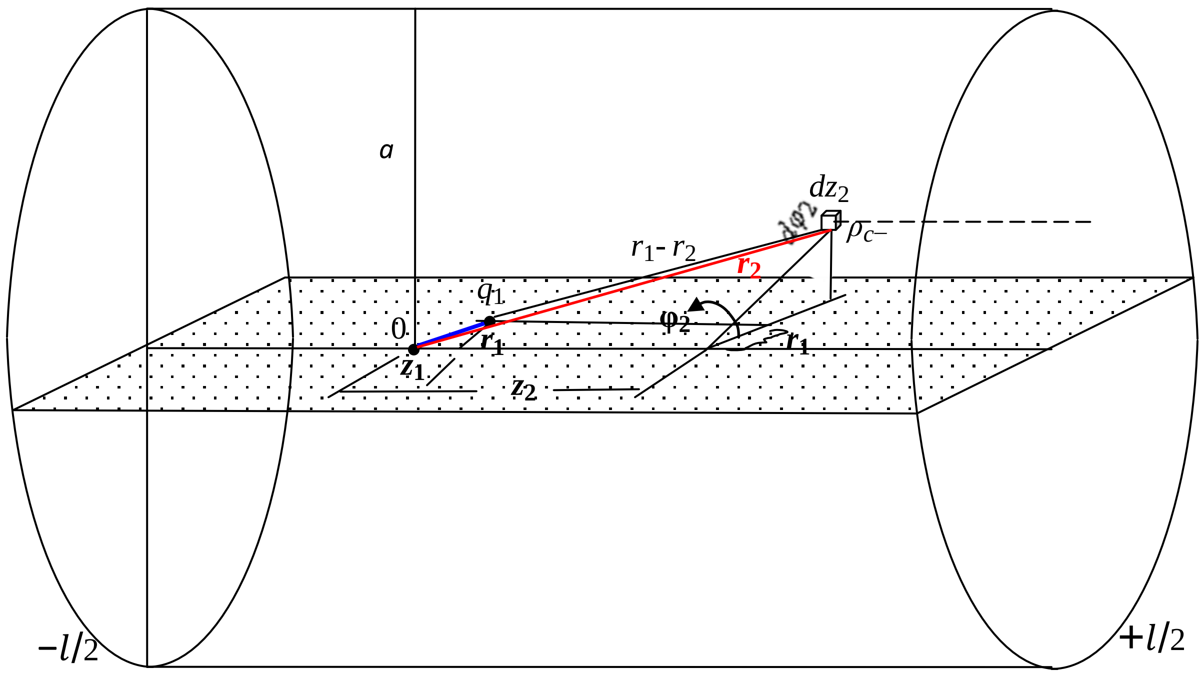


Fig. 3 (not to scale) Conduction charge element dq_{c-} located at $\vec{r}_2$, moving with velocity $\vec{v}_{2-} = v_{2-}(z_2, t)\hat{z}$ and acceleration $\vec{a}_{2-} = a_{2-}(z_2, t)\hat{z}$.

As a first approximation we assume that the charge density of the conduction electrons is equal and opposite to the charge density of the positive lattice, namely

$$\rho_{c-} = -\rho_+ \tag{1}$$

In order to integrate these three forces we utilize Weber's electrodynamics. According to Weber's force equation, the force exerted by a charge element dq_2 located at $\vec{r}_2$, moving with velocity $\vec{v}_2$and acceleration $\vec{a}_2$ on a point charge q_1 located at $\vec{r}_1$, moving with velocity $\vec{v}_1$ and acceleration $\vec{v}_1$ is given by Eq. 3.24 [1, Chapter 3] and it would be instructive for the reader to derive this equation using Eqs. 3.13 and 3.14[1]

$$d\vec{F} = \frac{q_1 dq_2}{4\pi\varepsilon_o}\frac{\hat{r}_{12}}{r_{12}^2}\left[1 + \frac{1}{c^2}\left(\vec{v}_{12}.\vec{v}_{12} - \frac{3}{2}(\vec{r}_{12}.\vec{v}_{12})^2 + \vec{r}_{12}.a_{12}\right)\right] \tag{2}$$

where ε_o= 8.85 × 10^{-12} $C^2 \cdot N^{-1} \cdot m^{-2}$ is the permittivity of vacuum, c = 3 × 10^8 $m \cdot s^{-1}$, $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$, $\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$, $\vec{a}_{12} = \vec{a}_1 - \vec{a}_2$, $r_{12} = |\vec{r}_{12}|$ and $\hat{r}_{12} = \vec{r}_{12}/r_{12}$ is the unit vector pointing from 2 to 1.

After calculating the force on a generic test charge we consider it to be a conduction electron. With Newton's second law of motion $F = ma$ we get one equation with two unknowns, the current and the density of free electricity (excess charge). The other equation connecting these two unknowns is that for the conservation of charges. With these two relations we then obtain the equation describing the propagation of electromagnetic signals in wires according to Weber's electrodynamics.

Straight Wire

Consider the straight wire of radius a and length $l >> a$ as shown in Figs. 1 to 3. We suppose a symmetrical current density $\vec{J} = J(z,t)\hat{z}$, where the z axis has been chosen along the axis of the wire with $z = 0$ at its center (Figs. 1, 2 and 3 are not drawn to scale). We consider cylindrical coordinates (r, φ, z), with r being the distance of the charge to the axis of the wire and not to the origin of the coordinate system (we do not employ the usual notation 'ρ' in the cylindrical coordinates to avoid confusion with the charge density).

As pointed out above, our procedure will be to integrate Eq. (2) for the force acting on the test charge q_1 due to the coulombian, velocity and acceleration terms. The charges exerting the force will be the surface charges with density σ_f, the positive lattice with density ρ_+, and the conduction electrons with density ρ_{c-}. We begin with the force exerted by the free (or, excess) surface charges on the test charge.

Forces exerted by the free (or, excess) surface charges on the test charge

As we are considering only the symmetrical situation in which the surface current does not depend on φ, the same will happen with the free surface charge density: $\sigma_f = \sigma_f(z, t)$. Accordingly the force on the test charge q_1 cannot depend on its poloidal angle φ_1. To simplify the calculations without any loss of generality we consider it located at $\varphi_1 = 0$, so that $\vec{r}_1 = r_1\hat{x} + z_1\hat{z}$, with velocity $\vec{v}_1 = \dot{x}_1\hat{x} + \dot{y}_1\hat{y} + \dot{z}_1\hat{z}$ and acceleration $\vec{a}_1 = \ddot{x}_1\hat{x} + \ddot{y}_1\hat{y} + \ddot{z}_1\hat{z}$.

Instead of integrating directly the coulombian force it is easier to integrate the electric potential and then obtain the force by taking the gradient of this potential as was done by Kirchhoff.

Force on test charge as gradient of potential

The formula that Kirchhoff or Weber would have used is $\vec{F} = -q\nabla V$ based on Action-at-a-distance principles (no field). This was the approach employed by Kirchhoff and we follow it here. (Recollect from the beginning of Appendix A that the electric field $\vec{E}$ (in Newtons/Coulomb) is $\vec{E} = \frac{\vec{F}}{q}$ where $\vec{F}$ is the force acting on charge q. Since the electric field is the gradient of potential V i.e. ($\vec{E} = -\nabla V$), then $\vec{F} = -q\nabla V$).

The coulombian potential $\varnothing = (r_1, z_1, t)$ where this test charge is located, due to the free (or, excess) surface charges in the current carrying wire, is then given by (with $dq_2 = \sigma_f\ a\ d\varphi_2\ dz_2$ located at $\vec{r}_2 = a\ cos\varphi_2\hat{x} + a\ sin\varphi_2\hat{y} + z_2\hat{z}$);

$$\varnothing(r_1, z_1, t)$$

$$= \frac{1}{4\pi\varepsilon_o}\int_{\varphi_2=0}^{2\pi}\int_{z_2=-l/2}^{l/2}\frac{\sigma_f(z_2,t)\ a\ d\varphi_2\ dz_2}{\sqrt{r_1^2+a^2-2r_1 a\ cos\varphi_2+(z_2-z_1)^2}}$$

$$= \frac{1}{4\pi\varepsilon_o}\int_{\varphi_2=0}^{2\pi}\int_{z_2=-l/2}^{l/2}\frac{\sigma_f(z_2,t)\ a\ d\varphi_2\ dz_2}{\sqrt{(1-2(r_1/a)\ cos\varphi_2+(r_1^2/a^2))((z_2-z_1)/a)^2}} \qquad (3)$$

Defining

$s^2 \equiv 1-2(r_1/a)\ cos\varphi_2 + (r_1^2/a^2)$, $0 \le r_1 \le a$ and $u \equiv (z_2 - z_1)/a$ this can be written as

$$\varnothing(r_1, z_1, t) = \frac{a}{4\pi\varepsilon_o}\int_{\varphi_2=0}^{2\pi}\int_{u=-(l+2z_1)/2a}^{(l-2z_1)/2a}\frac{\sigma_f(au+z_1,t)\ d\varphi_2\ du}{\sqrt{s^2+u^2}}$$

Kirchhoff was able to solve these integrals utilizing the approximations

$$l >> a \qquad l >> r_1 \qquad \text{and} \qquad l \gg |z_1| \qquad (4)$$

The main ideas of Kirchhoff's approach are presented although not Kirchhoff's exact steps and derivation.

For any given r_1 and φ_2 the maximum value of $1/\sqrt{s^2+u^2}$ is at $u = 0$ i.e. $(z_2 - z_1)/a = 0$. For z_2 far from z_1 the value of u will be of the order $l/a >> 1$ due to the approximation (4). This means that $1/\sqrt{s^2+u^2}$ will be close to zero if z_2 is far from z_1, as s is of the order of unity. This is because from $s^2 \equiv 1-2(r_1/a)\ cos\varphi_2 + (r_1^2/a^2)$ with $0 \le r_1 \le a$ we have $0 \le s^2 \le 1$.

Because when z_2 is far from z_1, the integrand is close to zero, it can be neglected. But, the integrand has a large value when $z_2 = z_1$. Making this brilliant deduction for an approximation, Kirchhoff could remove $\sigma_f(au + z_1, t)$ from the integrand by taking its value at $z_2 = z_1$, because whatever the value of σ_f at other locations, it will not affect the integrand significantly for reasons cited above.

We are then led to the approximate result

$$\varnothing(r_1, z_1, t) = \frac{a\sigma_f(z_1,t)}{4\pi\varepsilon_o}\int_{\varphi_2=0}^{2\pi}\int_{u=-(l+2z_1)/2a}^{(l-2z_1)/2a}\frac{d\varphi_2\ du}{\sqrt{s^2+u^2}} \qquad (5)$$

We label the double integral I and with the approximation $|z_1| \ll l$:

$$I \equiv \int_{\varphi_2=0}^{2\pi} \int_{u=-l/2a}^{l/2a} \frac{d\varphi_2\, du}{\sqrt{s^2+u^2}} \qquad (6)$$

where $s^2 \equiv 1 - 2(r_1/a)\ \cos\varphi_2 + (r_1^2/a^2)$.

Integration with respect to *u* and putting the above and lower limits yields (refer steps here "Integration of 1/ sqrt (x^2+a^2)dx" https://www.youtube.com/watch?v=MWfA85-Mb50)

$$I = \int_{\varphi_2=0}^{2\pi} d\varphi_2 \ln \frac{\sqrt{s^2+(l/2a)^2}+(l/2a)}{\sqrt{s^2+(l/2a)^2}-(l/2a)} \qquad (7)$$

From the definition of s^2 and $0 \le r_1 \le a$, we can write

$$0 \le s^2 \le 1.$$

According to Eq. (4)

$$l >> a \qquad l >> r_1 \qquad \text{and} \qquad l \gg |z_1|$$

From these last two equations $l/2a >> s^2$, so that the numerator of Eq. (7) becomes

$$\sqrt{s^2+(l/2a)^2} + (l/2a) \approx (l/2a + l/2a) = l/a.$$

The denominator of Eq. (7) becomes analogously

$$\sqrt{s^2+(l/2a)^2} - (l/2a) = \sqrt{(l/2a)^2(1+(s/(l/2a))^2} - (l/2a)$$

Then, using Maclaurin's series expansion (Refer "Maclaurin Series of Sqrt(1+x) " https://www.emathzone.com/tutorials/calculus/maclaurin-series-of-sqrt1x.html)

this becomes

$$(l/2a)\sqrt{(1+(s/(l/2a))^2} - (l/2a) \approx (l/2a)\left(1+\frac{1}{2}\left(\frac{s}{l/2a}\right)^2\right) - (l/2a).$$

Therefore, Eq. (7) is

$$I \equiv \int_{\varphi_2=0}^{2\pi} d\varphi_2 \ln \frac{\sqrt{s^2+(l/2a)^2}+(l/2a)}{\sqrt{s^2+(l/2a)^2}-(l/2a)} = \int_{\varphi_2=0}^{2\pi} d\varphi_2 \ln \frac{(l/a)}{\frac{(l/2a)}{2}\left(\frac{s}{l/2a}\right)^2} = \int_{\varphi_2=0}^{2\pi} d\varphi_2 \ln \frac{(2l/a)(l/2a)}{s^2}$$

$$= \int_{\varphi_2=0}^{2\pi} d\varphi_2 \ln \frac{(l/a)^2}{s^2} = \int_{\varphi_2=0}^{2\pi} d\varphi_2 [\ln(l/a)^2 - \ln(s^2)]$$

$$= \int_{\varphi_2=0}^{2\pi} [\ln(l/a)^2] d\varphi_2 - \int_{\varphi_2=0}^{2\pi} [\ln(s)^2] d\varphi_2$$

$$= \ln(l/a)^2 |\varphi_2|_0^{2\pi} - \int_{\varphi_2=0}^{2\pi} \left[\ln\left(1 - 2\frac{r_1}{a}\ \cos\varphi_2 + \frac{r_1^2}{a^2}\right)\right] d\varphi_2$$

$$= 4\pi \ln\frac{l}{a} - \int_{\varphi_2=0}^{2\pi}\left[\ln\left(1 - 2\frac{r_1}{a}\cos\varphi_2 + \frac{r_1^2}{a^2}\right)\right]d\varphi_2 \tag{8}$$

This last integral (in Eq. (8)) is equal to zero if $r_1 \le a$. If $r_1 > a$, we can put r_1^2/a^2 in evidence and utilize once more this result to solve the last integral, namely:

$$\int_{\varphi_2=0}^{2\pi}\left[\ln\left(1 - 2\frac{r_1}{a}\cos\varphi_2 + \frac{r_1^2}{a^2}\right)\right]d\varphi_2 = 0 \qquad \text{if } r_1 \le a \tag{9}$$

for a test charge located in the body of the wire

$$\int_{\varphi_2=0}^{2\pi}\left[\ln\left(1 - 2\frac{r_1}{a}\cos\varphi_2 + \frac{r_1^2}{a^2}\right)\right]d\varphi_2 = 2\pi\ln\frac{r_1^2}{a^2} \qquad \text{if } r_1 \ge a \tag{10}$$

for a test charge located outside the body of the wire. **Note:** Eqn (9) can be verified numerically using Mathematical softwares by setting limits $\varphi_2 = 10^{-7}$ to $2\pi - 10^{-7}$ and giving several values of $\frac{r_1}{a} < 1$ to show the result is null. With this result, Eqn (10) follows putting r_1^2/a^2 in evidence. This means that the final value of the integral *I* defined by Eq. (6) is found to be

$$I = 4\pi\ln\frac{l}{a} \qquad \text{if } r_1 \le a \tag{11}$$

$$I = 4\pi\ln\frac{l}{r_1} \qquad \text{if } r_1 \ge a \tag{12}$$

Therefore, we can write the solution to Eq. (5) as

$$\varnothing(r_1, z_1, t) = \frac{a\sigma_f(z_1,t)}{\varepsilon_o}\ln\frac{l}{a} \qquad \text{if } r_1 \le a \tag{13}$$

$$\varnothing(r_1, z_1, t) = \frac{a\sigma_f(z_1,t)}{\varepsilon_o}\ln\frac{l}{r_1} \qquad \text{if } r_1 \ge a \tag{14}$$

The coulombian force is then given by ($\vec{F} = -q\nabla V$, see "Force on test charge as gradient of potential" above)

$$\vec{F} = -\frac{q_1 a}{\varepsilon_o}\frac{\partial\sigma_f}{\partial z_1}\left(\ln\frac{l}{a}\right)\hat{z} \qquad \text{if } r_1 < a \tag{15}$$

$$\vec{F} = \frac{q_1 a\,\sigma_f}{\varepsilon_o}\frac{\hat{r}_1}{r_1} - \frac{q_1 a}{\varepsilon_o}\frac{\partial\sigma_f}{\partial z_1}\left(\ln\frac{l}{r_1}\right)\hat{z} \qquad \text{if } r_1 \ge a \tag{16}$$

Forces on the moving test charge

Later on we consider the contribution to this force due to the motion of the test charge and of the free surface charges showing that they are negligible. These last two equations (Eqs. 15 and 16) are then the expressions for the force on the test charge due to the free surface charges.

Forces exerted by the positive stationary lattice and the conduction electrons on the test charge

We turn now to the force on the test charge due to the positive stationary lattice and to the conduction electrons. As the lattice is at rest we have $\vec{v}_{2+} = 0$ and $\vec{a}_{2+} = 0$. According to Weber's Force equation (Eq. (2)), the force of the lattice of Fig. 2 on the test charge q_1 is then given by:

$$\vec{F} = \int_{r_2=0}^{a} \int_{\varphi_2=0}^{2\pi} \int_{z_2=-l/2}^{l/2} \frac{q_1\, dq_{2+}}{4\pi\varepsilon_o} \frac{\hat{r}_{12}}{r_{12}^2} \left[1 + \frac{1}{c^2}\left(\vec{v}_1 \cdot \vec{v}_1 - \frac{3}{2}(\hat{r}_{12} \cdot \vec{v}_1)^2 + \vec{r}_{12} \cdot \vec{a}_1\right)\right] \tag{17}$$

Before integrating this force we consider the force on the test charge due to the moving conduction electrons Fig. 3 (with velocity $\vec{v}_{2-}$ and acceleration $\vec{a}_{2-}$). From Eq. 2 this is given by:

$$\vec{F} = \int_{r_2=0}^{a} \int_{\varphi_2=0}^{2\pi} \int_{z_2=-l/2}^{l/2} \frac{q_1\, dq_{c-}}{4\pi\varepsilon_o} \frac{\hat{r}_{12}}{r_{12}^2}$$

$$\times \left[1 + \frac{1}{c^2}\left(\vec{v}_{12-} \cdot \vec{v}_{12-} - \frac{3}{2}(\hat{r}_{12} \cdot \vec{v}_{12-})^2 + \vec{r}_{12} \cdot \vec{a}_{12-}\right)\right] \tag{18}$$

We need to integrate these forces over the volume of the wire of Figs. 2 and 3. To this end we replace dq_{2+} by $\rho_+\, r_2\, d\varphi_2\, dz_2\, dr_2$ and dq_{c-} by $\rho_{c-}\, r_2\, d\varphi_2\, dz_2\, dr_2$. Due to our approximation Eq. (1) that the wire is essentially neutral, except for the surface charges considered above, we can then add Eqs. (17) and (18) with consideration that the forces of the lattice and the conduction electron are opposite in direction on the test charge.

In the equations (17) and (18), the relational quantities of direction, velocity and acceleration viz. $\vec{r}_{12}$, $\vec{v}_{12}$ and $\vec{a}_{12}$ are expanded using the relations in the para below Eq. (2) before addition.

The coulombian term and the terms proportional to v_1^2, to $(\hat{r}_{12} \cdot \vec{v}_1)^2$ and to $\vec{r}_{12} \cdot \vec{a}_1$ cancel out. The remaining terms are (with $c^2 = 1/4\pi\varepsilon_o$ and taking out of the integral the constant ρ_{c-}):

$$\vec{F} = \frac{\mu_o q_1\, \rho_{c-}}{4\pi} \int_{r_2=0}^{a} \int_{\varphi_2=0}^{2\pi} \int_{z_2=-l/2}^{l/2} \frac{\hat{r}_{12}}{r_{12}^2} \Big[v_{2-}^2 - 2\vec{v}_1 \cdot \vec{v}_2 + 3(\hat{r}_{12} \cdot \vec{v}_1)(\hat{r}_{12} \cdot \vec{v}_{2-}) -$$

$$\frac{3}{2}(\hat{r}_{12} \cdot \vec{v}_{2-})^2 - \vec{r}_{12} \cdot \vec{a}_{2-}\Big]\, r_2\, d\varphi_2\, dz_2\, dr_2 \tag{19}$$

The terms proportional to v_{2-}^2 and to $(\hat{r}_{12} \cdot \vec{v}_{2-})^2$ are usually small compared to the coulombian forces Eqs. (15) and (16). Moreover, they point towards the radial direction $\hat{r}_1$ [10]. The radial electric field is also called *motional* electric field and is predicted by Weber's law. It is not predicted by classical electromagnetic theory. As we are interested only in the longitudinal propagation of the signal we will neglect these terms.

The terms depending on $\vec{v}_1 \cdot \vec{v}_{2-}$ and $(\hat{r}_{12} \cdot \vec{v}_1)(\hat{r}_{12} \cdot \vec{v}_{2-})$ give rise to the magnetic force $q_1\vec{v}_1 \times \vec{B}$, [1, Secs. 6.6 and 7.4] and [11,12]. The reader should note that the magnetic field *B* in the references cited is an "external" magnetic field *B* while the magnetic field *B* due to the velocity squared terms $\vec{v}_1 \cdot \vec{v}_{2-}$ and $(\hat{r}_{12} \cdot \vec{v}_1)(\hat{r}_{12} \cdot \vec{v}_{2-})$ is an internal magnetic field. This should not be deemed to violate the rule that a current carrying wire cannot exert a force on itself (see Chapter 6 of "Fundamentals of electric theory and circuits" below Fig. 6), because Weber's law complies with Newton's third law and the sum of ALL internal forces will always go to zero. The reader is advised to check the derivation which gives rise to the magnetic field in Weber's force law in the section "Derivation of Ampere's Force law from Weber's Force Law" above and the derivation of the field *B* from the velocity while making note of the role of Eq. 4.23[1, Chapter 4] in converting the velocities multiplied by charge elements into current elements.

As the current is in the longitudinal direction $\hat{z}$, the magnetic field will be in the poloidal direction $\hat{\varphi}$. The test charge considered here will be a conduction electron moving in the $\hat{z}$ direction, so that $q_1\vec{v}_1 \times \vec{B}$ will be in the radial direction $\hat{r}_1$. As we are interested only in the longitudinal propagation of the signal along the *z* direction, we will not consider these terms either.

We then need to take into account the acceleration term. As we are considering a straight wire with $\vec{v}_{2-} = v_{2-}(z_2,t)\hat{z}$ and $\vec{a}_{2-} = a_{2-}(z_2,t)\hat{z}$, this term will appear when there is alteration of the strength of the current (acceleration of the conduction electrons).

With $\vec{r}_1 = r_1\hat{x} + z_1\hat{z}$ and $\vec{r}_2 = r_2\, cos\varphi_2\hat{x} + r_2\, sin\varphi_2\hat{y} + z_2\hat{z}$ this term can be written as

$$\vec{F} = \frac{\mu_o q_1\, \rho_{c-}}{4\pi} \int_{r_2=0}^{a} \int_{\varphi_2=0}^{2\pi} \int_{z_2=-l/2}^{l/2} r_2\, d\varphi_2\, dz_2\, dr_2$$

$$\times \frac{(r_1 - r_2 cos\varphi_2)\,\hat{x} - r_2 sin\varphi_2\hat{y} - (z_2 - z_1)\,\hat{z}}{\left[r_1^2 + r_2^2 - 2r_1r_2cos\varphi_2 + (z_2 - z_1)^2\right]^{3/2}}\ (z_2 - z_1)\, a_{2-}(z_2,t) \qquad (20)$$

Integrating in φ_2 the *y* component goes to zero. Once more with Eq. (4) and Kirchhoff 's great idea of approximation we remove $a_{2-}(z_2,t)$ from the integrand taking its value at $z_2 = z_1$, yielding

$$\vec{F} = \frac{\mu_o q_1\, \rho_{c-}\, a_{2-}(z_1,t)}{4\pi} \int_{r_2=0}^{a} \int_{\varphi_2=0}^{2\pi} \int_{z_2=-l/2}^{l/2} r_2\, d\varphi_2\, dz_2\, dr_2$$

$$\times \frac{(r_1 - r_2 cos\varphi_2)\,\hat{x} - (z_2 - z_1)\,\hat{z}}{\left[r_1^2 + r_2^2 - 2r_1r_2cos\varphi_2 + (z_2 - z_1)^2\right]^{3/2}}\ (z_2 - z_1) \qquad (21)$$

Integrating in z_2 the *x* component goes to zero, as we are supposing $l >> |z_1|$.

Calling $z_2 - z_1 \equiv m$ and $r_1^2 + r_2^2 - 2r_1r_2cos\varphi_2 \equiv n^2$ we are then led to:

$$\vec{F} = \frac{\mu_o q_1 \, \rho_{c-} \, a_{2-}(z_1,t)}{4\pi} \hat{z} \int_{r_2=0}^{a} \int_{\varphi_2=0}^{2\pi} \int_{m=-l/2}^{l/2} r_2 \, d\varphi_2 dm \, dr_2 \frac{m^2}{(n^2+m^2)^{3/2}} \qquad (22)$$

These integrals can be solved utilizing the approximation in Eq. (4).

Let $J \equiv \int_{r_2=0}^{a} \int_{\varphi_2=0}^{2\pi} \int_{m=-l/2}^{l/2} \frac{m^2 dm}{(n^2+m^2)^{3/2}} \, r_2 \, d\varphi_2 \, dr_2$ (23)

where $n^2 \equiv r_1^2 + r_2^2 - 2r_1 r_2 cos\varphi_2$.

The indefinite integral in *m* (refer "Evaluate integrate x^2/(x^2 + a^2)^(3/2) dx " https://www.youtube.com/watch?v=is_KZjs58c0) yields

$$-\frac{m}{\sqrt{n^2+m^2}} + \ln(\sqrt{n^2+m^2} + m)$$

From approximation Eq. (4) and taking the two limits of the integral in *m* we are led to

$$J = \int_{r_2=0}^{a} \int_{\varphi_2=0}^{2\pi} (-2 + \ln \frac{l^2}{n^2}) \, r_2 \, d\varphi_2 \, dr_2$$

$$= 2\pi a^2 (\ln l - 1) - \int_{r_2=0}^{a} \int_{\varphi_2=0}^{2\pi} \ln(r_1^2 + r_2^2 - 2r_1 r_2 cos\varphi_2) \, r_2 \, d\varphi_2 \, dr_2$$

(24)

From Eqs. (9) and (10) we can solve this last integral, yielding $2\pi a^2 \ln a + \pi(r_1^2 - a^2)$ if $r_1 \le a$ or $2\pi a^2 \ln r_1$ if $r_1 \ge a$. Utilizing once more the approximation (4) we are then led to:

$$J = 2\pi a^2 \ln \frac{l}{a} \qquad \text{if } r_1 \le a \qquad (25)$$

$$J = 2\pi a^2 \ln \frac{l}{r_1} \qquad \text{if } r_1 \ge a \qquad (26)$$

Therefore, the solution of Eq. (22) is

$$\vec{F} = -\frac{q_1 \mu_o a^2 \, \rho_{c-} \, a_{2-}(z_1,t)}{2} (\ln \frac{l}{a}) \hat{z} \qquad \text{if } r_1 \le a \qquad (27)$$

$$\vec{F} = -\frac{q_1 \mu_o a^2 \, \rho_{c-} \, a_{2-}(z_1,t)}{2} (\ln \frac{l}{r_1}) \hat{z} \qquad \text{if } r_1 \ge a \qquad (28)$$

These equations can be written as

$$\vec{F} = -q_1 \frac{\partial \vec{A}}{\partial t} \quad (29)$$

where

$$\vec{A}(r_1, z_1, t) = \frac{\mu_o}{2\pi} I(z_1, t)(\ln\frac{l}{a})\hat{z} \qquad \text{if } r_1 \le a \quad (30)$$

$$\vec{A}(r_1, z_1, t) = \frac{\mu_o}{2\pi} I(z_1, t)(\ln\frac{l}{r_1})\hat{z} \qquad \text{if } r_1 \ge a \quad (31)$$

and

$$I(z_1, t) = \pi a^2 \rho_{c-} v_{2-}(z_1, t) \quad (32)$$

$$\frac{\partial I}{\partial t} = \pi a^2 \rho_{c-} a_{2-}(z_1, t) \quad (33)$$

Here $I(z_1, t)$ is the total current through the cross section πa^2 in $z = z_1$, at the time *t*.

The force of free (or excess) charges due to their acceleration on the test charge

Up to now we included the forces on a test charge due to the free electricity (or excess charges) considered at rest and to the motion of the conduction electrons. We might think that the free electricity is moving together with the conduction electrons, so that we would need to calculate the force of this free electricity on a test charge taking into account the acceleration of σ_f. If this is done, we obtain essentially Eqs. (27) and (28) with σ_f replacing $a\rho_{c-}/2$.

The density of conduction electrons in a typical metallic conductor is of the order of one electron per atom, yielding: $|\rho_{c-}| \approx 10^{10}\ \mathrm{C \cdot m^{-3}}$. We can estimate σ_f observing that in linear conductors it is a linear function of the axial coordinate[8]. For instance, consider a coaxial cable of inner radius *a* and outer radius *b*, with conductivity *g*. Then the surface charge density of the inner conductor σ_f^a when it is flowing a current *I* is given by (see Sommerfeld's Electrodynamics[6, pp. 125-130]): $\sigma_f^a = -\frac{\varepsilon_o I z}{\pi g a^3} \ln(b/a)$. With a copper wire of inner radius 1 mm, outer radius 2 mm, carrying a current of 100 A the charge density at the large distance of 100 m is only $|\sigma_f^a| \approx 10^{-7}$ C/m^2. We then have $\sigma_f^a \approx 10^{-7}$ C/m^2 << $a\rho_{c-}/2 \approx 10^{7}$ C/m^2.

This means that in these calculations it does not matter if this free electricity is moving or not with the conduction electrons. The effect of their motion is negligible compared with the effect of the moving conduction electrons. We can then say that all relevant electromagnetic effects have been taken into account here.

The frictional (or resistive) force of the moving test charge

We now suppose the test charge to be a conduction electron: $q_1 = -e = -1.6 \times 10^{-19}$ C, $\vec{v}_1 = v_{2-}(z_1, t)\hat{z}$ and $\vec{a}_1 = a_{2-}(z_1, t)\hat{z}$. In this case we must also include the frictional force due to its collisions with the lattice (see "Working definition of Current" in Section 1.18 Visualizing Current and Neutrality in Conductors in "Fundamentals of Electric Theory and Circuits"). The average value of this force can be represented by $-b\vec{v}_1$, where the coefficient of friction b is given by $b = \rho_+ e/g = -e\rho_{c-}/g$, g being the conductivity of the wire[13, Sec. 7.7 and 14, Introduction]. Writing the resistance R of the wire of radius a as $R = l/g\pi a^2$ this can also be written as $b = -e\rho_{c-}\pi a^2 R/l$.

We can now write down the z component of the equation of motion for a conduction electron applying Newton's second law of motion $F_z = ma_z$. Considering the frictional force plus Eqs. (15), (27) and dropping the subscript 1 yields:

$$\underbrace{\frac{ea}{\varepsilon_o}\left(\ln\frac{l}{a}\right)\frac{\partial\sigma_f(z,t)}{\partial z}}_{\text{Eq. 15 (force of free (excess) surface charges)}} + \underbrace{\frac{e\mu_o a^2\rho_{c-}}{2}\left(\ln\frac{l}{a}\right)a_{2-}(z,t)}_{\text{Eq. 27 (combined force of lattice ions and conduction electrons, a neutral system)}}$$

$$+\underbrace{\frac{e\pi a^2 R\rho_{c-}}{l}v_{2-}(z,t)}_{\text{Frictional force due to collision of conduction electrons with the lattice}} = \underbrace{ma_{2-}(z,t)}_{\text{Acceleration of the test charge (conduction electron) of mass } m \text{ due to the combined effect of all the forces on the LHS}} \qquad (34)$$

Usually $|e\mu_o a^2\rho_{c-}(\ln(l/a))/2| \gg m$ [14] so that we can neglect the term ma_{2-} in this equation. For instance, for a one meter wire with one millimeter diameter we have, with $e = 1.6 \times 10^{-19}$ C, and $\rho_{c-} \approx -10^{10}$ C·m^{-3}, $|e\mu_o a^2\rho_{c-}(\ln(l/a))/2| \approx 2\times10^{-21}$ kg, which is much greater than the electron mass $m = 9 \times 10^{-31}$ kg.

With Eqs. (32) and (33) this equation can then be written as (multiplying it by $\varepsilon_o/ea\ \ln(l/a)$ and utilizing $c^2 = 1/\mu_o\varepsilon_o$):

$$\frac{\partial\sigma_f}{\partial z} + \frac{1}{2\pi a}\frac{1}{c^2}\frac{\partial I}{\partial t} = -\frac{\varepsilon_o R}{a\, l \ln(l/\mathrm{a})}I \qquad (35)$$

There are two unknowns in this equation, σ_f and I. In order to relate them we utilize the equation for the conservation of charges, $\nabla\cdot\vec{J} = -\partial\rho_f/\partial t$. For the case considered here of a current flowing in the z direction over the cross section πa^2 of the wire of radius a, this is equivalent to:

$$\frac{\partial I}{\partial z} = -2\pi a \ \frac{\partial \sigma_f}{\partial t} \qquad (36)$$

See "The continuity equation for the surface current of a cylindrical conductor" in Section 1.

Applying $\partial/\partial t$ in Eq. (35), multiplying it by $-2\pi a$ and utilizing Eq. (36) yields:

$$\frac{\partial^2 I}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 I}{\partial t^2} = \frac{2\pi\varepsilon_o R}{l \ln(l/a)} \ \frac{\partial I}{\partial t} \qquad (37)$$

This is the equation of telegraphy, which will also be satisfied by σ_f, by $\emptyset(a, z, t)$ and by the z component of $\vec{A}$ at $r = a$, $A_z(a, z, t)$.

If the resistance of the wire is negligible, Weber's electrodynamics plus Newton's second law of motion predicts a current flow obeying the wave equation. That is, with a signal propagating at light velocity.

Comparison of the motion of em signals in wires with transverse vibrations of strings

Consider a string of mass M stretched between two points at $x = 0$ and $x = L$ as shown in the Fig.4.

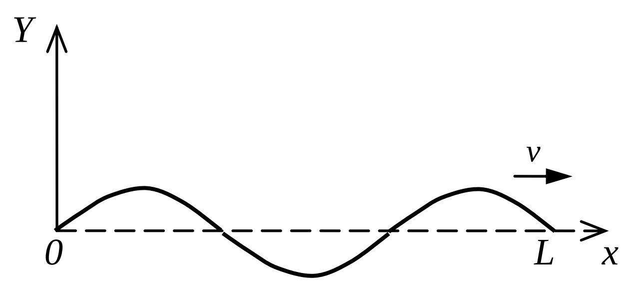


Fig. 4 The string is pulled upward and released.

Consider that the string with initial tension T is stretched by an amount ΔL by pulling it up and then released. Let the lateral displacement be given by $Y(x,t)$.

The equation of transverse vibration of the string is

$$\frac{1}{v^2}\frac{\partial^2 Y}{\partial t^2} = \frac{\partial^2 Y}{\partial x^2} \qquad (38)$$

where $v = \sqrt{\frac{T}{M/L}}$. is the velocity of the traveling wave on the string. The molecules do not move from left to right. They merely move up and down creating the illusion of a wave traveling.

A comparison of Eqs. (38) and (37) prompted Kirchhoff to write in his paper Poggendorff's Annalen (1857) No.2 translated into English and published in the Phil. Mag. S.4.Vol. 13 No. 88, June 1857 that the motion of electricity in wires is quite similar to the propagation of a wave [transverse vibrations] in a tended [taut] wire.

Weber's Potential Energy

Weber presented a formulation of the potential energy for charged particle interactions. It was the first example of a force between charges which depended not only on the distance between them but *also on their velocities*.

$$U \equiv \frac{q_i q_j}{4\pi\varepsilon_o}\frac{1}{r_{ij}}\left(1-\frac{\dot{r}_{ij}^2}{2c^2}\right)$$

The first term of this energy is the usual Coulombian potential energy. The second term is a mixture of kinetic and potential energies because it depends not only on the distance between the charges but also on their velocities(1, Section 3.3).

Difficulties with the Maxwell-Lorentz electrodynamics (Classical electrodynamics)

According to Maxwell-Lorentz electrodynamics, the force on an element of volume d^3r at $\vec{r}$ containing a charge and current density ρ and $\vec{J}$ is given by the Lorentz force

$$\underset{\text{Total force}}{\frac{d^3\vec{F}_{\mathrm{M}}}{d^3r}} = \underset{\text{Coulombic}}{-\rho\nabla\Phi} \underset{\text{electromotive}}{-\frac{\rho}{c}\frac{\partial\vec{A}}{\partial t}} \underset{\text{Lorentz (magnetic)}}{+\vec{J}\times\left(\nabla\times\vec{A}\right)} \qquad (39)$$

where Φ and $\vec{A}$, are the scalar and (magnetic) vector potentials.

In terms of moving charges, the Maxwell-Lorentz force on a charge q with velocity $\vec{v}$ at $\vec{r}$ due to a charge q' with velocity $\vec{v}'$ at $\vec{r}'$ is given by

$$c^2\vec{F}_{\mathrm{M}} = -qq'\left[\frac{c^2\vec{R}}{R^3}-\frac{1}{R}\frac{d\vec{v}'}{dt}-\frac{1}{R^3}(\vec{v}\cdot\vec{v}')\vec{R}+\frac{1}{R^3}\left(\vec{R}\cdot\vec{v}\right)\vec{v}'\right] \qquad (40)$$

where $\vec{R}$ is the line joining the two charges [15].

It may be seen from the second and fourth terms on the right of Eq. (40) that the Maxwell-Lorentz force between two point charges violates Newton's third law; as these forces do not act along the line $\vec{R}$ joining the two charges, and interchanging primes and unprimes does not yield merely a change in sign. It should be remarked that a failure to obey Newton's third law is a very serious matter; as it implies impractical consequences and actions, such as the violation of the conservation of energy, ability to propel a space craft using only forces internal to the space craft itself, and the ability to lift oneself by one's own boot straps. Even a casual glance at Eq. (40) is, thus, sufficient to show that the Maxwell theory cannot be based solely upon the forces between isolated point charges, in contrast to the Weber theory. In addition, Eq. (40) does not agree with the experimental evidence which will not be described here but details are available [15, Sections 4, 5, 6, 7 and 8].

Limitations of Maxwell theory [15]

The Maxwell theory, being incapable of prescribing the correct force between two moving point charges, cannot be regarded as a fundamental theory. The special situations and limiting conditions under which the Maxwell theory works are

1) The interaction between moving point charges must not be involved.

2) Macroscopic quantities of material and macroscopic distributions of charge must always be assumed.

3) A source must be confined to a finite volume, and it must vanish on the surface of this volume.

4) A detector must be confined to a finite volume, where source and detector do not occupy the same volume.

5) Source currents must form closed current loops so that $\nabla \cdot \vec{A} = 0$.

6) The force on an accelerating charge or time varying current due to a static charge distribution must not be involved.

7) Induction must be limited to closed current loops due to the net time rate of change of the magnetic flux through the loop.

8) Induction in only a portion of a closed loop cannot be involved.

9) Induction in open circuits cannot be involved.

In contrast, the Weber theory, being a fundamental theory based upon the interaction between two moving charges, appears to have no limitations at all.

It maybe shown[1] that Weber's Force can be used to obtain Ampere's force between current elements, Gauss's Law, Lorentz's force and Faraday's law.

Using Weber's electrodynamics to obtain Maxwell's equations[1]

When using Weber's Electrodynamics to obtain the full set of Maxwell's equations the following should be borne in mind:

1) Gauss's law [1, Section 3.2]: $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_o}$. In applying Weber's Force to obtain Gauss's Law, Assis [1, Chapter 8, Section 8.1] mentions that the proof is valid only when there is no motion between the charges. If charge motion is assumed, Weber's force predicts a motional electric field [10] which violates Gauss's Law.

2) Non-existence of magnetic monopole [1, Section 4.7]: $\nabla \cdot \vec{B} = 0$. This equation can be derived from Weber's force through the integrated form of Ampere's force (force of a closed circuit on a current element) only when there is charge neutrality of the current elements and does not take into account the presence of say, surface charges. Secondly, in Ampere's force the circuit was assumed to be closed and did not take into account currents in open circuits as in the case of the charge and discharge of a capacitor.

3) Faraday's Law [1, Section 5.3]: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$. Again in deriving Faraday's law from Weber's Force or potential energy [1, Section 5.3], it was assumed that the circuits were closed and is not valid for open circuits.

4) Magnetic circuital law [1, Section 4.7] or Ampere-Maxwell law (Chapters 5, Section 5.22): $\nabla \times \vec{B} = \mu_o \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$. Remember that Faraday's Law and the Ampere-Maxwell law are both necessary for the propagation of electromagnetic waves in space (Chapter 10, Sections 10.2 and 10.10). In the derivation [1, Section 4.7] of this equation from Weber's force, it was supposed that the circuits were stationary and the distance between charges did not depend on time.

According to J. P. Wesley [15], Maxwell's electrodynamic theory is a generalization of slowly varying effects and yet does not correctly predict a few effects. For example, Maxwell's electrodynamics fails to explain the force on Ampere's bridge, the tension required to rupture current carrying wires, the force to drive the Graneau-Hering submarine, the force to drive the oscillations in a current carrying mercury wedge [15], and unipolar induction [16,17]. Also see Fig. 3.4, Section 3.2, "Contradictions in traditional teaching", in the article "A qualitative guide to electricity" by Hermann Härtel in the folder "Teacher's Guide" in the CD alongwith the book.

These effects are explained satisfactorily by Weber's electrodynamics, the details of which are available [15,16 and 17].

Weber's Electrodynamics in Terms of Fields and application to radiation phenomena

While Weber's electrodynamics is an action–at–a–distance theory, Weber's force law includes the quantity $c = \frac{1}{\sqrt{\varepsilon_o \mu_o}}$ and Weber was able to measure it and found its value to be equal to light velocity. So, while the interaction is supposed to be instantaneous, each of the interacting charges has inertia so that their reaction to the applied force (acquired velocities and accelerations, etc.) is a function of their inertial masses.

Let us not forget that Newton derived the propagation of sound at a finite velocity using action-at-a-distance mechanics and d'Alembert showed the propagation with a finite velocity of perturbations along a stretched string by its molecules using action–at–a–distance theory.

These systems are many bodied and the interactions are instantaneous of the bodies having inertia. Thus, though the interaction of any two particles may be considered to be instantaneous, the collective behavior (macroscopic wave, etc.) has a finite characteristic velocity as was shown in the derivation of Eq. 37. In principle therefore it should be possible to derive a finite velocity of light waves and electromagnetic waves with an action–at–a–distance theory.

While Weber's Force is based on action–at–a–distance theory (without a field), an electromagnetic field appropriate to Weber's Force can be derived and this field may be extended to rapidly varying effects and radiation by introducing time retardation.

The Maxwell field theory displaced the Weber action–at–a–distance theory toward the end of the last century; because the Maxwell theory predicted Hertz electromagnetic waves, and the Weber theory could not [16]. The failures of the Maxwell theory for slowly varying effects (to explain the Ampere bridge for example) are seldom mentioned in textbooks.

An action–at–a–distance theory can be represented directly in terms of the force between two particles, such as Weber's force

$$\vec{F}_{ji}^{W} = -\frac{q_i q_j}{4\pi\varepsilon_o}\frac{\vec{r}_{ij}}{r_{ij}^2}\left[1 + \frac{1}{c^2}\left(\vec{v}_{ij}\cdot\vec{v}_{ij}\right) - \frac{3}{2}\left(\hat{r}_{ij}\cdot\vec{v}_{ij}\right)^2 + \vec{r}_{ij}\cdot\vec{a}_{ij}\right] \qquad (41)$$

or it can be represented in terms of intermediate fields. In the field representation a particle, or distribution of particles, is viewed as first giving rise to an intermediate field [15]. It is then the field that acts on another particle thereby giving rise to the observed force. Although these two representations may evoke different images of physical mechanisms involved; they are, in fact, mathematically isomorphic (when no time retardation is involved).

The derivation proposed by J. P. Wesley [(16)] uses notation that may be confusing and Assis has rewritten the principal features using modern vector notation in Weber's Electrodynamics[(1, Chapter 8 Section 8.3)]. J. P. Wesley [(16)] began with Weber's force equation

$$\vec{F}_{ji}^{W} = -\frac{q_i q_j}{4\pi\varepsilon_o}\frac{\vec{r}_{ij}}{r_{ij}^2}\left[1 + \frac{1}{c^2}\left(\vec{v}_{ij}\cdot\vec{v}_{ij}\right) - \frac{3}{2}\left(\hat{r}_{ij}\cdot\vec{v}_{ij}\right)^2 + \vec{r}_{ij}\cdot\vec{a}_{ij}\right] \tag{42}$$

and rewrote it by replacing q by ρdV and $\rho\vec{v}$ by $\vec{J}$ and then neglected the velocity squared forces to yield

$$\frac{d^6\vec{F}_{ji}}{dV_i dV_j} = \frac{\hat{r}_{ij}}{4\pi\varepsilon_o r_{ij}^2}[\rho_i\rho_j - \frac{2\vec{J}_i\cdot\vec{J}_j}{c^2} + \frac{3(\hat{r}_{ij}.\vec{J}_i)(\hat{r}_{ij}.\vec{J}_j)}{c^2} + \frac{\rho_j\vec{r}_{ij}}{c^2}.\frac{\partial\vec{J}_i}{\partial t} - \frac{\rho_i\vec{r}_{ij}}{c^2}.\frac{\partial\vec{J}_j}{\partial t}] \tag{43}$$

After integrating the resulting force equation over a fixed volume *Vj* he obtained

$$\frac{d^3\vec{F}_{ji}}{dV_i} = -\rho_i\nabla\Phi + \vec{J}_i\times(\nabla\times\vec{A}) - \rho_i\frac{\partial\vec{A}}{\partial t} - \vec{J}_i\nabla\cdot\vec{A} + \frac{\Phi}{c^2}\frac{\partial\vec{J}_i}{\partial t} + (\vec{J}_i\cdot\nabla)\nabla\Gamma + \rho_i\nabla\frac{\partial\Gamma}{\partial t} - \left[\left(\frac{\partial\vec{J}_i}{\partial t}\right)\cdot\nabla\right]\frac{\vec{G}}{c^2} \tag{44}$$

where Φ and $\vec{A}$ are the usual electric (coulomb) and magnetic vector potentials and Γ and $\vec{G}$ are two new potentials. The operators ∇_1(gradient) and $\nabla_1\times$ (cross-product) are to be applied at the position of charge 1, while Φ_2 and $\vec{A}_2$ are the potentials due to the charges in volume V_2.

For full details and the definitions of the potentials Γ and $\vec{G}$, the reader may refer to the Chapter 8, Section 8.3 in Weber's Electrodynamics[(1)].

The usual electric and magnetic fields are obtained from $\vec{E} = -\nabla\Phi$ and $\vec{B} = \nabla\times\vec{A}$.

Once having expressed Weber electrodynamics in terms of fields, it may be immediately extended to rapidly varying effects and electromagnetic radiation by introducing time retardation. This is done by replacing time in the equations for the potentials by the retarded time $t^* = t - R/c$ where $R = |\vec{r} - \vec{r}'|$ is the distance between the charges. Note: There have been attempts to introduce time retardation between two particles without an intermediate field, but these attempts have not been successful [(15)].

Certainly the Weber field theory is more complicated with its two additional field potentials Γ and $\vec{G}$ in Eq. (44); so one might expect it to provide some advantages. A problem of predicting the self-torque on the Pappas-Vaughan Z-antenna provides an excellent test case [(16)]. The Weber field theory predicts a zero self torque; whereas the Maxwell field theory (classical electromagnetic theory) predicts a sizeable nonvanishing self torque. However, results of an experiment by Pappas and Vaughan showed that there is zero self-torque which clearly indicates the correctness of the Weber theory for rapidly varying time retarded fields compared with the Maxwell theory.

But then where are the particles or charges or many-bodied medium in a vacuum? *That could enable waves of electromagnetic fields to propagate*?

Newer theories including photons may provide an answer and the task to explain radiation phenomena as observed in antennae and radio communication using Weber's electrodynamics whether in its original form or with modifications still remains to be done.

For the present however, fields predicted by Maxwell's field theory are sufficiently accurate for the purpose of analysis of wave propagation and the design of antennas and waveguides.

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