Fundamentals of Electric Theory and Circuits

by

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Electric Potential of a Set of Point Charges

The Electric Potential

Just as we do not associate a direction with a temperature at a point anywhere, likewise we do not associate a direction to electric potential *V*. It is defined at a point $\vec{r}$ by

$$V(\vec{r}) = -\int_{ref}^{\vec{r}} \vec{E}.d\vec{l}$$ where *ref* is some standard reference point.

The minus sign in the equation for potential will make the potential at points around a positive charge to be positive if the reference *ref* is taken to be at infinity and whose potential is assigned to be zero, as shown below.

The electric field at a point which is distance *r* from a positive charge *q* located at the origin is given by $\vec{E} = \frac{1}{4\pi\,\varepsilon_o}\frac{q}{r^2}$.

The potential at *r* is then $V(r) = -\int_{\infty}^{r} \vec{E}\,dr = -\int_{\infty}^{r} \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}dr$

$$= \frac{-q}{4\pi\varepsilon_o}\int_{\infty}^{r}\frac{dr}{r^2} = \frac{q}{4\pi\varepsilon_o}\frac{1}{r}\bigg|_{\infty}^{r} = \frac{q}{4\pi\varepsilon_o}\frac{1}{r}$$

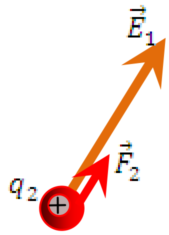
which is positive.

The reader may have noted that the definition makes use of the electric field which is an entity of classical electromagnetism due to Maxwell and Lorentz in the 1860s. Also see the definitions in Refs. [8] and [42].

The electric field can be defined by

$$\vec{F}_2 = q_2 \vec{E}_1$$

The equation says that the force on particle 2 is determined by the charge of particle 2 and by the electric field (which can be depicted using vectors) made by *all other charged particles in the vicinity*, as shown.



Prior to 1860, the known quantities of electromagnetic theory were the force between charges due to Coulomb (1785) and the potential function as defined by Kirchhoff (1845). See Ref. [38] concerning the field descriptions, their anomalies.

For an example of how Kirchhoff showed that signals propagate in wires without the concept of fields, see “Charge Densities and Continuity and Prop of em signals in wires” in the pdf_files in the CD. Coulomb, Ampere, Kirchhoff and Weber worked with charged particle interactions without a field like the gravitational interaction between bodies. The force between charge particles can be directly obtained from $\vec{F} = -q\nabla V$ **without an electric field.**

It was due to **Laplace’s and Faraday’s** ideas that the object and particle interactions began to be ***modeled* using gravitational and electric and magnetic ‘fields’**. In electricity and magnetism the electric and magnetic fields were formulated in the 1860s by Maxwell and Lorentz and their representation as *field lines*, *lines of force* due to Faraday and as *tubes of force* are due to J. J. Thomson and Maxwell.

There are several interpretations given to the electric and magnetic fields among which we have adopted one that models them as an “alteration of space” in this text and also attributed to the fields properties like their penetration of matter, filling the entire universe and that they are invisible (see Sections 1.4, 1.11 and 2.28).

These concepts of fields in Classical Electromagnetism and computations of their strengths are useful in the analysis of circuit processes and in wave propagation and the reader should not dismiss these outright. Computations using field quantities are sufficiently accurate and precise in most applications of circuit theory though the idea as to how the fields themselves carry momentum is still the subject of study by physicists.

Obtaining the field E from V

To get back the field $\vec{E}$(which has a direction) from *V* (which does not have a direction) we use

$\vec{E} = -\nabla V = -gradient\ V$ with the stipulation that $\vec{E}$ *points in the maximum* increase of *V* and field is given by $\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}$.

In general the potential will change from point to point in space. The quantity ∇*V* is a vector which points, at each point in space, in the direction of the *largest increase* in *V* around this point. Positive charges in a region of variable potential will move from the larger to the smaller potential if they are not under the influence of other forces (that is, they will move in the same direction given by $\vec{E}$). Negative charges will move in the opposite direction. Also see Refs. [8] and [42].

Electric Potential of a Set of Point Charges

The potential V_r at a distance r from a point charge Q is given by

$$V_r = k\frac{Q}{r} = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r}$$

The voltage in space around a charge **Q** is negative if **Q** is negative and positive if **Q** is positive. **The sign of the charge is important in calculating the potential of a set of charges.** The potential is a scalar quantity, not a vector.

Thus, the potential of a group of charges is simply the mathematical sum of potentials due to each charge.

$$V = V_1 + V_2 + \ldots$$

In two-dimensions, the electric potential **V**(***x***,***y***) can be pictured as a surface where the potential is plotted along the ***z***-axis. A visualization of the potential around a set of point charges is better afforded by equipotential contours.

Equipotentials

An equipotential surface is a surface upon which the potential has the same value at all points on that surface. In two-dimensional plots, equipotentials form closed lines as shown in Fig. 1.

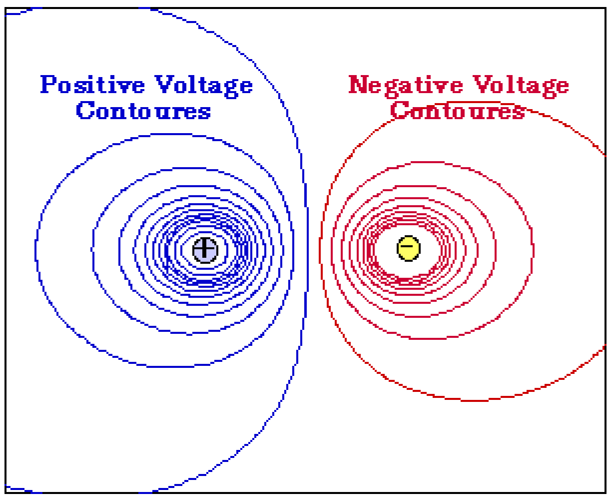


Fig. 1 voltage contours around a pair of charges near each other.

Image courtesy: Richard Vawter, Western Washington University

We can also draw the contours in a 3-D pictorial representation since the electric field of point charges exist in all directions in space and the lines that depict the field can be drawn in all directions from the charge. Such a contour diagram of potential of a positive and negative point charge pair separated by a small distance is shown in Fig. 2.

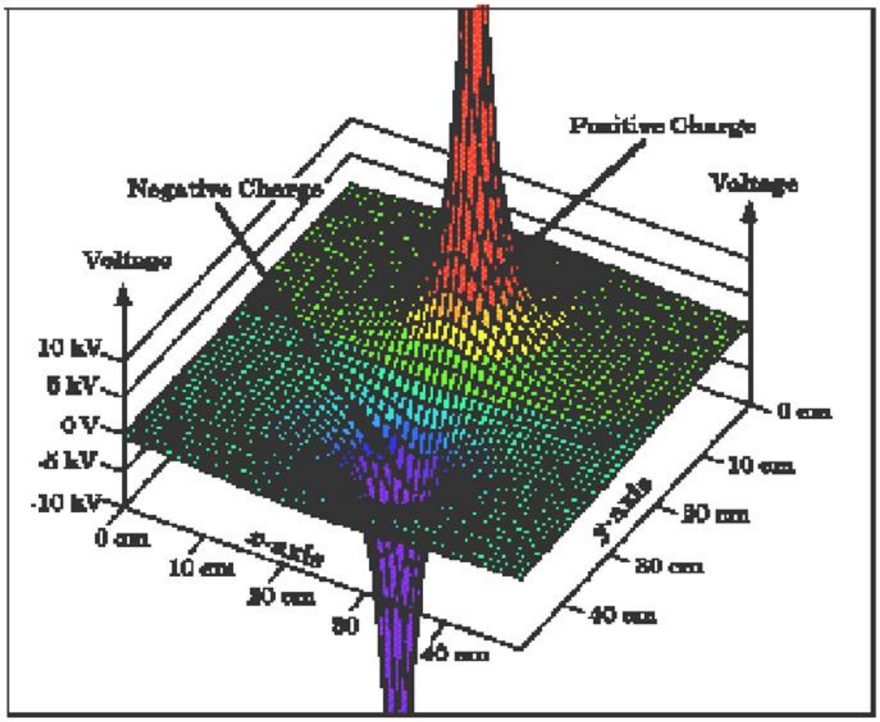


Fig. 2 The voltage contours of a positive and negative point charge pair
Image courtesy: Richard Vawter, Western Washington University

The electric potential **V**(**x,y**) can be pictured as a surface where the potential is plotted along the **z**-axis (Fig.2). Horizontal slices at equal distance of potential along the vertical axis of the potential produce equipotential contours when viewed from above along the axis of the potential as shown in Fig. 1.
Note the large potential values (ranging in kilovolts) close to the charges in Fig. 2. There is also a region between the point charges where the potential zero.

It is possible to find regions of zero potential in the presence of electric field and regions where there exists a potential but the electric field is zero as solutions to the following numerical problem will illustrate.

Potential of Two Point Charges

Two point charges q_1 = -88.0 nC and q_2 = 360 nC are located at the fixed positions at x_1 = 22.0 cm and x_2 = 96.0 cm on the x-axis.

(A) What is the electric potential due to both charges at x = 50.0 cm?

(B) Where on the x-axis is the electric potential due to both charges equal to zero?

Solution

A) Find **V(x = 50 cm).**

Since 50 cm is in between the two charges the electric potential must be written as

$$V(x) = k\frac{q_1}{r_1} + k\frac{q_2}{r_2}$$
$$= k\frac{q_1}{x - x_1} + k\frac{q_2}{x_2 - x}$$

if $\mathbf{r_1}$ and $\mathbf{r_2}$ are to be positive numbers.

Substituting the values of

$$V(.50\,\text{m}) = k\frac{q_1}{.50\,\text{m} - x_1} + k\frac{q_2}{x_2 - .50\,\text{m}}$$
$$= 9\times10^9\,\frac{-88.0\times10^9\,\text{C}}{.50\,\text{m} - .22\,\text{m}} + 9\times10^9\,\frac{360\times10^9\,\text{C}}{.96\,\text{m} - .50\,\text{m}}$$
$$= -2828.5\,\text{V} + 7043.4\,\text{V} = 42149\,\text{V} = \underline{4.21\,\text{kV}}$$

This is quite a large value of potential, and is expected considering the point charges are in the range of nanocoulombs. An electron has a charge of only 1.6×10^{-19} coulombs, which is a tiny charge in comparison with the amount of charge given in the problem.

(B) Find the points $\mathbf{x_o}$ on the x-axis were **V** = 0.

The potential immediately surrounding a positive charge is positive while the potential immediately surrounding a negative charge is negative. Somewhere in between the two charges the two potentials must add to up to zero. Since this region is greater than x_1 and less than x_2, then the electric potential of two point charges in between the charges is

$$V(x) = k\frac{q_1}{r_1} + k\frac{q_2}{r_2}$$
$$= k\frac{q_1}{x - x_1} + k\frac{q_2}{x_2 - x}$$

if r_1 and r_2 are to be positive. Setting **V(x_o)** equal to zero and solving for x_o,

$$
\begin{aligned}
V(x_o) &= k\frac{q_1}{x_o - x_1} + k\frac{q_2}{x_2 - x_o} \\
0 &= k\frac{q_1}{x_o - x_1} + k\frac{q_2}{x_2 - x_o} \\
\frac{q_1}{x_o - x_1} &= -\frac{q_2}{x_2 - x_o} = \frac{q_2}{x_o - x_2} \\
q_1(x_0 - x_2) &= q_2(x_o - x_1) \\
x_o(q_1 - q_2) &= q_1 x_2 - q_2 x_1 \\
x_o &= \frac{q_1 x_2 - q_2 x_1}{q_1 - q_2} \\
&= \frac{(-88\times 10^{9}\,\mathrm{C})(.96\,\mathrm{m}) - (360\times 10^{9}\,\mathrm{C})(.22\,\mathrm{m})}{-88\times 10^{9}\,\mathrm{C} - 360\times 10^{9}\,\mathrm{C}} \\
&= .36536\,\mathrm{m} = \underline{36.5\,\mathrm{cm}}
\end{aligned}
$$

You could check this answer by placing it in the equation for the potential to see if the potential is zero at this point.

However, an incorrect equation for **V(*x*)** is written, then the answer would check, but that answer would be incorrect. For example, if you wrote

$$V(x_o) = k\frac{q_1}{x_o - x_1} + k\frac{q_2}{x_o - x_2}$$

and solved for x_o you would find that x_o = -1.9412 cm and V(-.019412) = 0. What is wrong is that this last equation is only valid if $x_o > x_2 > x_1$ and this solution is inconsistent with that assumption. Because of sign cancellation, this is actually another zero of the potential obtained incorrectly.

Note that if $|q_1| > |q_2|$ then there would be a valid zero point beyond the second charge.

The best way to see this would be to plot the potential **V(*x*)** as a function of ***x*.**

Fig. 3 shows the contour of potential of each charge and the total potential as the sum of the potential due to both the charges in the space around the charges.

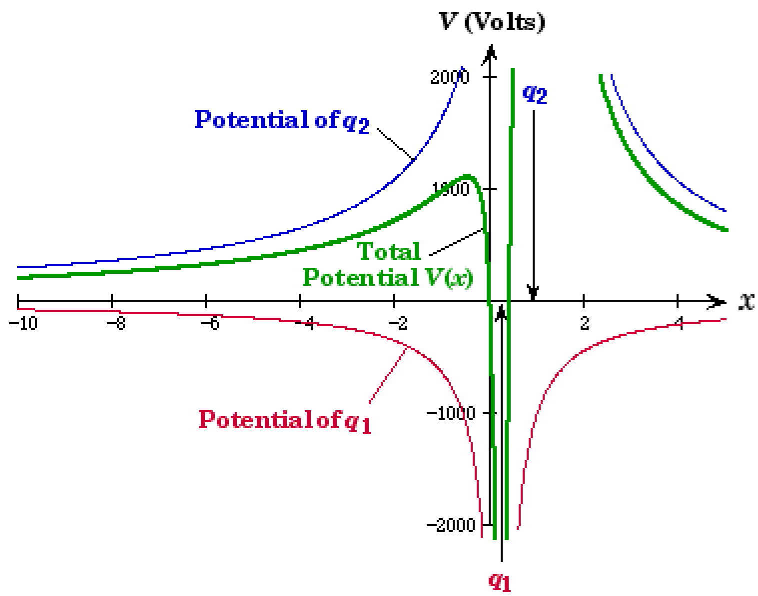


Fig. 3 Potential of individual charges q_1 and q_2 and the total potential as the sum of potential of q_1 and q_2

Image courtesy: Richard Vawter, Western Washington University

We can see that there is another solution just to left of the first charge $\boldsymbol{q}_1$ somewhere near zero.

One might reasonably argue that the second, larger charge which is positive has some positive potential in the region around the location of the negative charge. Since the positive charge is larger than the negative charge, the potential of the larger charge could be larger in absolute size in some region to the left of the negative charge.

To the left of the negative charge the total potential will be,

$$V(x) = k\frac{q_1}{r_1} + k\frac{q_2}{r_2}$$
$$= k\frac{q_1}{x_1 - x} + k\frac{q_2}{x_2 - x}$$

to make $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ both to be positive. We can solve for the location of the zero of the potential in a manner similar to that above.

$$V(x_o) = k\frac{q_1}{x_1 - x_o} + k\frac{q_2}{x_2 - x_o}$$

$$0 = k\frac{q_1}{x_1 - x_o} + k\frac{q_2}{x_2 - x_o}$$

$$\frac{q_1}{x_1 - x_o} = -\frac{q_2}{x_2 - x_o} = \frac{q_2}{x_o - x_2}$$

$$q_1(x_0 - x_2) = q_2(x_1 - x_o)$$

$$x_o(q_1 + q_2) = q_1 x_2 + q_2 x_1$$

$$x_o = \frac{q_1 x_2 + q_2 x_1}{q_1 + q_2}$$

$$= \frac{(-88 \times 10^{9}\,\mathrm{C})(.96\,\mathrm{m}) + (360 \times 10^{9}\,\mathrm{C})(.22\,\mathrm{m})}{-88 \times 10^{9}\,\mathrm{C} + 360 \times 10^{9}\,\mathrm{C}}$$

$$= -.01941\,\mathrm{m} = \underline{-1.94\,\mathrm{cm}}$$

A closer view of the potential (Fig. 4) is even more interesting.

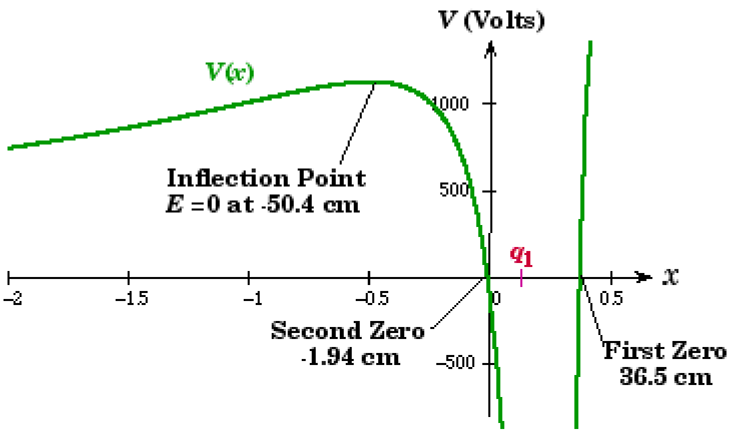


Fig. 4 Closer view of the potential contours of the pair of opposite point charges q_1 and q_2.

Image courtesy: Richard Vawter, Western Washington University

The inflection point is the location where the *slope of* the potential (and *not* the potential) is zero. Since the slope of the potential is equal to the electric field, $\mathbf{E_x} = -\,\mathbf{dV/d}x$, the electric field is equal to zero at this local maximum.

This means that if you place a third charge of either sign at this location it will experience no force even though the potential is over a thousand volts.

Potential with zero Electric field

It may sound strange that there is a point or maybe a small patch of region in space that has a potential with zero electric field. This is not difficult to understand if one were to give a little thought to the definition of potential.

Recollect from Section 2.20 the answer to our question “Where the voltage is”, that potential is merely a number that indicates the amount of work done to push a test charge from a point at infinity to a target point whose potential is to be determined. We should visualize the field configurations in regions of space because what nature provides is the electric field and not potential.

The target point may be located in a region where the electric field is the resultant of several point charges. The regions through which the test charge was pushed maybe filled with a field that was varying in magnitude and direction. Yet, the target point maybe located in a region where the resultant of the fields due to the source point charges is zero; but, that doesn’t qualify to nullify the work already done in pushing the test charge to that point through field filled regions. Hence, the finite value for potential even in regions where the field is zero.